Absorbing boundary conditions for wave equations, curvature and finite elements

Jukka Tuomela

UNITÉ DE RECHERCHE
INRIA-ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tél.:(1) 39 63 55 11

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Jukka TUOMELA

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Conditions aux limites absorbantes pour l'équation des ondes, courbure et éléments finis

Jukka Tuomela

INRIA, Rocquencourt
BP 105
78153 Le Chesnay Cedex
France

Helsinki University of Technology
02150 Espoo
Finland

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Abstract

We first review some stability questions for hyperbolic equations and theory of pseudo-differential operators which are relevant when constructing stable high order absorbing boundary conditions. In particular we point out some problems which seem to be open. Then in the numerical part we present some test cases which show the importance of the curvature term which has usually been somewhat neglected.

Resumé

Nous donnons d'abord un aperçu de certains aspects des problèmes de stabilité pour les équations hyperboliques et de la théorie des opérateurs pseudodifferentiels dont on a besoin pour construire des conditions aux limites absorbantes d'ordre élevé et stables. Nous soulevons quelques questions qui semblent être ouverts. Ensuite, nous présentons des résultats des simulations numériques qui mettent en évidence l'importance du terme de courbure qui a été souvent un peu négligé.

1 Introduction

In this work we consider matters which are related to the absorbing boundary conditions for the wave equation. This is generalization and extension of the work begun in [DT].

First we shall review rather thoroughly the derivation of absorbing boundary conditions using pseudodifferential operators. High order conditions then raise some difficult stability questions which still are not completely solved. We shall point out some difficulties that appear. One could take the point of view that most part of these sections are implicitly contained in [TA], [EM1], [EM2], [HT], [KR], [MO] and [DL8]. However, we thought it worthwhile to give and explicit exposition of these things because it seems that such an overview does not exist (but see [SE]).

Then we pass to the numerical treatment of the absorbing boundary conditions. Usually hyperbolic equations are solved with finite differences (though not always, see [BO], [UM] and [PM]), but we shall use standard P_1 finite elements. Consequently the discretization comes from the variational formulation which means that the boundary conditions are part of the general variational equation. This in turn has some implications on the practical implementation of the absorbing boundary conditions. In particular, we shall argue that taking into account the curvature of the boundary is in fact quite essential and give some numerical examples to support this claim. Sésques in [SE] has reached the same conclusion in a similar context.

Theoretically this curvature term has of course been known, but in the finite difference context where naturally rectangular domains are used it is not needed. On the other hand there appears the 'corner problem': corners of the rectangle have to treated separately. We shall show that the curvature term in fact also gives a natural solution to the corner problem (any corner, not just the right angled corner). This is nice from the practical point of view: absorbing boundary can have any shape and there are no special points.

The actual numerical simulations are done in two dimensions, but the principles remain the same in any dimension, so that the conclusions should be valid also in three dimensions.

2 Pseudodifferential Operators

2.1 Preliminaries

Let us start by considering the wave equation in one space dimension

$$u_{tt} - u_{xx} = 0$$

whose solutions can be written as $u(x,t) = f(x+t) + g(x-t)$, where f and g are 'arbitrary' functions. An obvious interpretation is to say that f (resp. g) represents a signal which travels to the left (resp. to the right). Let us take as our domain $\mathbb{R}_+ = \{ x \in \mathbb{R} \mid x > 0 \}$. Sooner or later f reaches the boundary and we would like it to 'pass through', that is to eliminate the reflections. Now if the signal just passes through then evidently it verifies the above equation even on the boundary, but the term u_{xx} would be very difficult (if not impossible) to treat, so one has to look for other solutions. The 'right' solution turns out to be to notice that $f(x+t)$ verifies also the simpler equation $u_t - u_x = 0$, and that gives a good boundary condition. The essential thing is that in one space dimension the wave operator can be factored as follows.

$$\partial_{tt} - \partial_{xx} = (\partial_t + \partial_x)(\partial_t - \partial_x) \tag{2.1}$$

So on the boundary we take the second factor of the operator. Similarly, to eliminate the signals travelling to the right we would take the first factor. Now denoting the normal derivative by ∂_n and using the convention that the normal is always directed to the exterior of the domain, it is seen that in both cases the boundary condition is $\partial_t u + \partial_n u = 0$. This solves the problem completely as every solution is a sum of left and right going waves. In more than one space dimension the situation is much more complicated, as we will shortly see, but the above factorization suitably generalized still gives the solution.

2.2 Constant Coefficients

Consider the two dimensional wave equation

$$u_{tt} - \Delta u = 0 \tag{2.2}$$

This has the plane wave solutions $u = \exp(i(k \cdot x - \omega t))$ where k is the wave vector and ω the (angular) frequency, provided that the dispersion relation $\omega^2 = |k|^2$ is satisfied. Remembering that k points to the direction of propagation, we see that left going plane wave satisfies the conditions $k_1 < 0$ and $\omega > 0$. Taking the domain to be $\mathbb{R}^2_+ = \{ x \in \mathbb{R}^2 \mid x_1 > 0 \}$, we then look for the boundary conditions which would let pass all the left going plane waves. As can easily be seen we cannot now factor the wave operator as in (2.1). Instead let us first take a Fourier transform of (2.2) with respect to x_2 and t, which gives

$$(-\tau^2 - \partial^2_{x_1} + \xi_2^2)\hat{u}(x_1,\xi_2,\tau) = (i\partial_{x_1} + \sqrt{\tau^2 - \xi_2^2})(i\partial_{x_1} - \sqrt{\tau^2 - \xi_2^2})\hat{u}(x_1,\xi_2,\tau)$$

We recall that the Fourier transform and its inverse in $\mathbb{R}^n$ are given by

$$\begin{aligned} \mathcal{F}(u(x)) = \hat{u}(\xi) &= (2\pi)^{-n} \int u(x) e^{-ix\cdot\xi} dx \\ \mathcal{F}^{-1}(\hat{u}(\xi)) = u(x) &= \int \hat{u}(\xi) e^{ix\cdot\xi} d\xi \end{aligned}$$

Although k and ω are formally the same as ξ and τ, we prefer for clarity to use different notation when we talk about the Fourier transform and plane waves. Then taking the second factor and applying (formally) the inverse Fourier transform one gets

$$i\,\frac{\partial u}{\partial x_1} - \int \sqrt{\tau^2 - \xi_2^2}\,\hat{u}\,e^{i(\xi_2 x_2 + \tau t)} d\xi_2\,d\tau = 0 \tag{2.3}$$

Now the plane wave being a tempered distribution has a Fourier transform which is $\mathcal{F}_{(x_2,t)}(\exp(i(k\cdot x - \omega t))) = \delta_{(k_2,-\omega)}\exp(i\,k_1 x_1)$, where $\delta_{(k_2,-\omega)}$ means the Dirac measure concentrated at the point $(\xi,\tau) = (k_2,-\omega)$. Substituting all this into (2.3), we see that the equation reduces to the dispersion relation, with the extra condition that k_1 be negative, and this is true for the left going plane waves. So we have some kind of solution to our problem, provided that we can give a meaning to the above integral operator. It looks like a pseudodifferential operator, but it is not one really. To see this, let us first give some definitions (all the material concerning the general properties of pseudodifferential operators can be found in [TA]).

Definition 1 *Let Ω be a domain in $\mathbb{R}^n$; m, $\rho \in \mathbb{R}$, with $0 \le \rho \le 1$. We say that $p \in C^\infty(\Omega \times \mathbb{R}^n)$ belongs to a symbol class $S_\rho^m(\Omega)$ if for any compact $K \subset \Omega$ and multi-indices α and β there exist a constant $C_{K,\alpha,\beta}$ such that*

$$\left|D_x^\beta D_\xi^\alpha p(x,\xi)\right| \le C_{K,\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|}$$

for all $x \in K$ and $\xi \in \mathbb{R}^n$.

Here $D_{x_j} = \partial_{x_j}/i$. Then we say that the operator $\mathcal{T}$ given by

$$\mathcal{T}u(x) = \int p(x,\xi)\hat{u}(\xi)e^{i\,x\cdot\xi}d\xi$$

is a pseudodifferential operator belonging to the operator class OS_ρ^m if its symbol $p \in S_\rho^m$. Note that if p is polynomial with respect to ξ then $\mathcal{T}$ is a differential operator which can be written as

$$\mathcal{T} = p(x,D) = \sum_{|\alpha|\le l} a_\alpha(x)D^\alpha$$

where a_α are arbitrary (smooth) functions. Evidently in this case $\mathcal{T} \in OS_1^l$. Then returning to the formula (2.3) it is obvious that the symbol $Q(\xi_2,\tau) = \sqrt{\tau^2 - \xi_2^2}$ does not belong to any symbol class S_ρ^m; it is not smooth when $|\tau| = |\xi_2|$ and this does not depend on the way we might extend it to the region $|\tau| < |\xi_2|$. However, it can be said to be a pseudodifferential operator in the following more general sense. Recall the characterization of the Sobolev spaces H^s by Fourier transform: $u \in H^s(\mathbb{R}^n)$ if and only if $(1+|\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)$. Now defining

$$Q(\xi_2,\tau) = \begin{cases} \sqrt{\tau^2-\xi_2^2} & \text{if } |\tau| \ge |\xi_2| \\ i\sqrt{\xi_2^2-\tau^2} & \text{if } |\tau| \le |\xi_2| \end{cases}$$

we obviously have

$$|Q(\xi_2,\tau)| \leq (1+\tau^2+\xi_2^2)^{1/2}$$

and consequently $\mathcal{T}_Q$ is a well defined operator from H^s to H^{s-1}.

However, the concept of pseudodifferential operator as in definition 1 will still be useful, as we will see in a moment. To this end let us consider wave packets. Intuitively a wave packet is a solution to the wave equation which is composed of plane waves all of which travel almost to the same direction. More precisely we might define it to be a solution of the form

$$u(x,t) = \int A(k,\omega)e^{i(x\cdot k-\omega t)}dk$$

where A (amplitude) has 'small' support. Of course k and ω satisfy the dispersion relation as before. The above integral is a kind of Fourier integral and so we can say that u is a left going wave packet if the support of its Fourier transform with respect to x_2 and t is contained in the set $B = \{\,(\xi_2,\tau) \in \mathbb{R}^2 \,|\, \tau - |\xi_2| \geq c > 0\,\}$. Now in this set Q is smooth and it can be smoothly extended to the whole of $\mathbb{R}^2$. This extension being quite arbitrary we can analyze Q in B and suppose that all its relevant properties remain the same outside B. Taking this point of view we can then easily verify that $Q \in S_1^1$ (note that Q is homogeneous of degree one, that is $Q(r\xi_2, r\tau) = rQ(\xi_2,\tau)$ where $r > 0$).

To get an idea of the nature of pseudodifferential operators we give the following

Theorem 1 *If the operator $\mathcal{T} \in OS_1^m$ and $1 < p < \infty$ then*

$$\mathcal{T} \;:\; W_p^s(\Omega)_{\text{comp}} \longrightarrow W_p^{s-m}(\Omega)_{\text{loc}}$$

Here the spaces W_p^s are the Sobolev spaces built with the usual L^p spaces, comp (resp. loc) refers to the functions which are compactly supported (resp. locally) in the corresponding space. Both comp and loc can be dropped if Ω is bounded and also if the symbol does not grow too wildly at infinity with respect to x (for instance if the symbol is constant outside some compact set).

The operator corresponding to the symbol Q (denoted by $\mathcal{T}_Q$) is still rather impractical: fixing some point $\bar{a} = (\bar{x}_1, \bar{x}_2, \bar{t})$, then the formula (2.3) tells us that to be able to calculate $\partial_{x_1}u$ at $\bar{a}$ we have to know $u(\bar{x}_1, x_2, t)$ for all x_2 and t; $\mathcal{T}_Q$ is non local in space and time. However, $\mathcal{T}_Q$ is a (strictly) hyperbolic operator in the sense that the following initial value problem is well defined

$$\begin{aligned} \frac{\partial u}{\partial x_1} + i\,\mathcal{T}_Q u &= 0 \\ u(0,x_2,t) &= g(x_2,t) \end{aligned}$$

where g is a given function. The direction x_1 has then after all become a time like direction as in one dimensional case.

The next step is then that we would like to approximate $\mathcal{T}_Q$ with a local operator. Now of course differential operators are local and also the converse is true: if a *linear* operator is local then it is a differential operator (see [DL3]). This means that one has to approximate Q by a polynomial or a rational function (with respect to Fourier variables). If $r = p_1/p_2$ is a rational function and $\mathcal{T}_r$ the corresponding operator where p_1 and p_2 polynomials, then we interpret the formula $v = \mathcal{T}_r u$ as

$$p_2(x, D)v = p_1(x, D)u \tag{2.4}$$

Then we have to decide what it means to approximate a pseudodifferential operator. Let us first give a definition.

Definition 2 *The symbol $p \in S_1^m$ admits an asymptotic expansion if there exists a sequence of symbols p_j which are homogeneous of degree j such that for every N*

$$p - \sum_{j=0}^{N} p_{m-j} \in S_1^{m-N-1}$$

So if we replace the symbol by its (truncated) asymptotic expansion the error will not necessarily be small but it will be smooth: the operator corresponding to their difference belongs to OS_1^{m-N-1}, so theorem 1 tells us that the error is less singular than the rest of the solution. This can be regarded as satisfactory if we consider that the solution is characterized by its singularities. In the same way we would like find a symbol R (a polynomial or a rational function) such that $Q - R \in S_1^l$ with $l < 1$. As such this is not possible: Q itself is homogeneous of degree 1 and if the same is true for R then of course it holds also for their difference. This shows that we cannot have an uniformly good approximation for all incidence angles. So let us consider the following subset of B: $\tilde{B} = \{ (\xi_2, \tau) \in B \mid |\xi_2| \leq \tilde{c} \}$. Other choices of $\tilde{B}$ are possible but the principle remains the same; this choice means that we want a good approximation at normal incidence. Let us then consider the following Taylor's expansions of Q.

$$Q = \sqrt{\tau^2 - \xi_2^2} \simeq \tau = R_1$$
$$Q = \sqrt{\tau^2 - \xi_2^2} \simeq \tau - \frac{\xi_2^2}{2\tau} = R_2$$

Then we see that in $\tilde{B}$ we have the following estimates.

$$\begin{aligned} |Q - R_1| &= \left| \sqrt{\tau^2 - \xi_2^2} - \tau \right| \\ &= \frac{\xi_2^2}{\sqrt{\tau^2 - \xi_2^2} + \tau} \leq \frac{C}{|\tau|} \end{aligned}$$

$$
\begin{aligned}
|\partial_\tau(Q-R_1)| &= \left|\frac{\tau}{\sqrt{\tau^2-\xi_2^2}}-1\right| \\
&= \left|\frac{\tau-\sqrt{\tau^2-\xi_2^2}}{\sqrt{\tau^2-\xi_2^2}}\right| \le \frac{C}{|\tau|^2}
\end{aligned}
$$

However, for all the other derivatives the estimates are exactly the same for Q and $Q-R_1$. Exactly in the same way we calculate that

$$
\begin{aligned}
|Q-R_2| &\le \frac{C}{|\tau|^3} \\
|\partial_\tau(Q-R_2)| &\le \frac{C}{|\tau|^4} \\
\left|\partial_\tau^2(Q-R_2)\right| &\le \frac{C}{|\tau|^5} \\
|\partial_{\xi_2}(Q-R_2)| &\le \frac{C}{|\tau|^3} \\
\left|\partial_{\xi_2}^2(Q-R_2)\right| &\le \frac{C}{|\tau|^3}
\end{aligned}
$$

Similarly all the other derivatives give at least the bound $|\partial^\alpha(Q-R_2)| \le C/|\tau|^3$, so we conclude that in the region $\tilde{B}$

$$
\begin{cases}
Q-R_1 \in S_0^{-1}\cap S_1^1 \\
Q-R_2 \in S_0^{-3}\cap S_1^1
\end{cases}
$$

Finally the differential operators and the boundary conditions corresponding to the symbols R_1 and R_2 are seen to be (remembering the interpretation (2.4))

$$
\begin{aligned}
&\frac{\partial u}{\partial x_1}-\frac{\partial u}{\partial t}=0 \\
&-\frac{\partial^2 u}{\partial x_1\partial t}+\frac{\partial^2 u}{\partial t^2}-\frac{1}{2}\frac{\partial^2 u}{\partial x_2^2}=0
\end{aligned}
$$

Now the derivative with respect to x_1 (x_2) is really a normal (tangential) derivative, so denoting the normal by n (tangent by s), we can write the above conditions for any boundary as follows

$$
\begin{aligned}
&\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial n}=0 \\
&\frac{\partial^2 u}{\partial t^2}+c\frac{\partial^2 u}{\partial n\partial t}-\frac{c^2}{2}\frac{\partial^2 u}{\partial s^2}=0
\end{aligned}
$$

Here c is the signal speed which up to now was supposed to be one. So the conditions make sense even when the boundary is curved, but soon we will see that in that case we should in fact add a supplementary term which depends on the curvature.

All the preceding considerations can immediately be generalized to the n dimensional case: all we have to do is to take the domain $\mathbb{R}_+^n = \{ x \in \mathbb{R}^n \,|\, x_1 > 0 \}$ and then everywhere in the previous section replace ξ_2^2 by $\xi_2^2 + \ldots + \xi_n^2$. This gives then directly the following absorbing boundary conditions

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial n} = 0$$
$$\frac{\partial^2 u}{\partial t^2} + c\frac{\partial^2 u}{\partial t \partial n} - \frac{c^2}{2}\tilde{\Delta} u = 0$$

where n denotes the normal derivative as before and $\tilde{\Delta}$ denotes the Laplace-Beltrami operator on the $n-1$ dimensional hypersurface.

2.3 Variable Coefficients and Curved Boundary

Let us start by defining the multiplication of two operators. In the previous section we saw that when the symbol is constant in the whole domain we can simply multiply the symbols and it gives the symbol of the product of operators. In general we have to use the following result.

Theorem 2 *Suppose $T_p \in OS_{\rho_1}^{m_1}$ and $T_q \in OS_{\rho_2}^{m_2}$ with symbols p and q respectively. Then $T_r = T_p T_q \in OS_\rho^{m_1+m_2}$ where $\rho = \min\{ \rho_1, \rho_2 \}$ and the symbol of T_r is*

$$r(x,\xi) \sim \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p(x,\xi)\, D_x^\alpha q(x,\xi)$$

We use the sign $\sim$ rather than equality because the right hand side is more like an asymptotic expansion than a convergent series, so that the best we can hope for is that the two sides differ by an 'smoothing operator', that is an operator of class $OS^{-\infty}$ (it belongs to OS_ρ^m for any m; in this case the parameter ρ does not make any difference). An example of such an operator is a convolution operator with C^∞ kernel.

Note that the multiplication is noncommutative and that evidently it reduces to the usual multiplication when q does not depend on x. The usual product gives, however, the principal part of the operator because we have

$$r(x,\xi) - p(x,\xi)\, q(x,\xi) \in S_\rho^{m_1+m_2-\rho_1}$$

Now to apply all this to the variable coefficient case let us consider the following differential operator

$$\mathcal{L}_I u = \rho(x,t)\partial_t^2 u - \nabla \cdot (\mu(x,t)\nabla u) \tag{2.5}$$

where μ could in general be a matrix valued function. To simplify the notations we will, however, suppose that μ is a (real valued) scalar function. The symbol of $\mathcal{L}$ is easily seen to be $l(x,t,\xi,\tau) = \xi \cdot \mu\xi - \rho\tau^2 - i(\nabla\mu)\cdot\xi$. We require that $\rho(x,t) \geq \rho_* > 0$ and $\mu(x,t) \geq \mu_* > 0$ for $\mathcal{L}_l$ to be a reasonable operator: these hypothesis guarantee that the corresponding initial value problem is well-posed. Now l is a second degree polynomial with respect to ξ and τ, so it could be factored simply as $(\xi_1 + g_1)(\xi_1 + g_2)$ where g_i are some functions of the rest of the variables. However, this does not give the factors of the operator. Instead we use the preceding theorem. Evidently $l \in S_1^2$ and its asymptotic expansion is simply given by

$$\begin{aligned} l_2(x,t,\xi,\tau) &= \mu(x,t)|\xi|^2 - \rho(x,t)\tau^2 \\ l_1(x,t,\xi,\tau) &= -i(\nabla\mu(x,t))\cdot\xi \\ l_j(x,t,\xi,\tau) &= 0 \qquad \text{if } j \leq 0 \end{aligned}$$

As in the previous section we then consider the domain $\mathbb{R}^n_+$ and as before we would like to construct a pseudodifferential operator $\mathcal{T}$ on the boundary such that

$$\mu\frac{\partial u}{\partial n} + i\mathcal{T}u = 0$$

Then considering the half space as before we can try the following factorization

$$\mathcal{L}_l = -\mathcal{L}_1\mathcal{L}_2 = -(\partial_{x_1} + i\mathcal{T}_p)(\mu\partial_{x_1} + i\mathcal{T}_q)$$

In terms of symbols this gives

$$l(x,t,\xi,\tau) = (\xi_1 + p(x,t,\tilde{\xi},\tau) \bullet (\mu\xi_1 + q(x,t,\tilde{\xi},\tau) \tag{2.6}$$

where $\bullet$ means that we take the 'symbol product' of theorem 2 and $\tilde{\xi} = \begin{pmatrix} \xi_2 & \ldots & \xi_n \end{pmatrix}$. Evidently we want that p and q are in $S_1^1(\mathbb{R}^{n-1})$. Next we naturally suppose that p and q admit asymptotic expansions

$$\begin{aligned} p(x,t,\tilde{\xi},\tau) &\sim \textstyle\sum_{j=0}^{\infty} p_{1-j}(x,t,\tilde{\xi},\tau) \\ q(x,t,\tilde{\xi},\tau) &\sim \textstyle\sum_{j=0}^{\infty} q_{1-j}(x,t,\tilde{\xi},\tau) \end{aligned} \tag{2.7}$$

Then we just substitute the expansions of p, q and l into (2.6) and then identify the terms having the same degree. In this way we can recursively calculate the p_j and q_j. Let us then see how pseudodifferential operators behave under a change of coordinates. Let Ω_1 and Ω_2 be two domains and let $\psi : \Omega_1 \to \Omega_2$ be a diffeomorphism. If $\mathcal{T}_{p_1}$ is an operator on Ω_1 we can define $\mathcal{T}_{p_2}$ on Ω_2 by $\mathcal{T}_{p_2}u(y) = \mathcal{T}_{p_1}u(\psi(x))$ where the right hand side is evaluated at $x = \psi^{-1}(y)$ ($x \in \Omega_1$ and $y \in \Omega_2$). We have then the following result.

Theorem 3 *If $T_{p_1} \in OS_1^m(\Omega_1)$ then $T_{p_2} \in OS_1^m(\Omega_2)$ and the symbol p_2 is given by*

$$p_2(y,\xi) \sim \sum_{\alpha\geq 0} \frac{1}{\alpha!}\varphi_\alpha(x,\xi) D_\xi^\alpha p_1(x, J^t(x)\xi)$$

As before the right hand side is evaluated at $x = \psi^{-1}(y)$; J is the Jacobian of ψ. As usual this operator is uniquely defined modulo a smoothing operator. Evidently if T is a differential operator then the above transformation reduces to the ordinary change of variables formula; in particular, in this case p_1 is a polynomial with respect to ξ, so that the above sum has only a finite number of terms. This result allows us to define the pseudodifferential operators on the manifold: just take Ω_2 as some coordinate neighborhood on the manifold and Ω_1 a suitable open set in $\mathbb{R}^n$. The functions φ_α are given by

$$\begin{aligned}
\varphi_\alpha(x,\xi) &= D_z^\alpha \exp(i\, d_x(z)\cdot\xi) \\
d_x(z) &= \psi(z) - \psi(x) - J(x)(z-x)
\end{aligned}$$

where in the definition of φ the right hand side is evaluated at $z = x$. Straightforward calculations then show that

$$\begin{aligned}
&\varphi_0(x,\xi) = 1 \\
&\varphi_\alpha(x,\xi) = 0 \qquad |\alpha| = 1 \\
&\varphi_\alpha(x,\xi) = i\, D_x^\alpha \psi(x)\cdot\xi \quad |\alpha| = 2 \text{ or } 3
\end{aligned}$$

After these generalities let us consider the problem (2.5) with $\rho = \mu = 1$ directly in n dimensional space taking the following domain: $\Omega = \{\, y \in \mathbb{R}^n \,|\, y_1 > f(\tilde{y})\,\}$, where f is some (positive) differentiable function and $\tilde{y} = (y_2, \ldots, y_n)$. Then Ω is diffeomorphic to the right half space $\mathbb{R}^n_+$ with the obvious diffeomorphism $\psi : \mathbb{R}^n_+ \to \Omega$ defined as follows

$$\psi : \begin{pmatrix} x_1 \\ \tilde{x} \end{pmatrix} \to \begin{pmatrix} y_1 \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} x_1 + f(\tilde{x}) \\ \tilde{x} \end{pmatrix}$$

Using this coordinate transformation we get the following operator in $\mathbb{R}^n_+$

$$\mathcal{L}_l u = u_{tt} - |\nabla f|^2 u_{x_1x_1} - \Delta u + 2\sum_{i=2}^{n} f_{x_i} u_{x_1x_i} + \Delta f u_{x_1} = 0$$

Note that this of the form (2.5) with μ a symmetric positive definite matrix and $\rho = 1$. Because f is a function of $\tilde{x}$ only we can either take the point of view that f is defined on $\mathbb{R}^{n-1}$ or $\mathbb{R}^n$ so ∇f is either $n-1$ or n vector. Then denoting $1 + |\nabla f|^2$ by a we see that the symbol l is given by

$$l(x,t,\xi,\tau) = -\tau^2 + a\xi_1^2 + \left|\tilde{\xi}\right|^2 - 2\xi_1 \nabla f \cdot \tilde{\xi} + i\,\Delta f \xi_1$$

Next we factor the above symbol as in (2.6) and look for the corresponding asymptotic expansions (2.7). Using the symbol product and starting with highest order terms we get the equation

$$-\tau^2 + a\xi_1^2 + \left|\tilde{\xi}\right|^2 - 2\xi_1 \nabla f \cdot \tilde{\xi} = a\xi_1^2 + \xi_1(ap_1 + q_1) + p_1 q_1$$

This leads to a system of equations

$$\begin{aligned} p_1 q_1 &= \left|\tilde{\xi}\right|^2 - \tau^2 \\ ap_1 + q_1 &= -2\nabla f \cdot \tilde{\xi} \end{aligned}$$

with the solutions

$$\begin{aligned} q_1 &= -\nabla f \cdot \tilde{\xi} \pm \sqrt{a(\tau^2 - \left|\tilde{\xi}\right|^2) + \left(\nabla f \cdot \tilde{\xi}\right)^2} \\ p_1 &= \frac{1}{a}(-\nabla f \cdot \tilde{\xi} \mp \sqrt{a(\tau^2 - \left|\tilde{\xi}\right|^2) + \left(\nabla f \cdot \tilde{\xi}\right)^2} \end{aligned}$$

Recall that we search for the operator for the left going wave and because the operator product is not commutative, this operator has to be the right factor of (2.6). Then comparing to the case where ψ is identity mapping one sees that the $-$ sign corresponds to the left going case, so we take q_1 with the $-$ sign and p_1 with the $+$ sign. Next we will calculate the terms p_0 and q_0. Applying the symbol product and collecting the terms of the appropriate degree we get the following equation

$$i\,\Delta f \xi_1 = \xi_1(ap_0 + q_0) + p_0 q_1 + p_1 q_0 - i\sum_{j=2}^{n} \frac{\partial p_1}{\partial \xi_j}(\xi_1 \frac{\partial a}{\partial x_j} + \frac{\partial q_1}{\partial x_j})$$

Then identifying the coefficients on both sides gives

$$\begin{aligned} ap_0 + q_0 - i\sum_{j=2}^{n} \frac{\partial p_1}{\partial \xi_j}\frac{\partial a}{\partial x_j} &= i\,\Delta f \\ p_0 q_1 + p_1 q_0 - i\sum_{j=2}^{n} \frac{\partial p_1}{\partial \xi_j}\frac{\partial q_1}{\partial x_j} &= 0 \end{aligned}$$

Eliminating p_0 leads to

$$i\,q_0(ap_1 - q_1) + \sum_{j=2}^{n} \frac{\partial p_1}{\partial \xi_j}(a\frac{\partial q_1}{\partial x_j} - q_1\frac{\partial a}{\partial x_j}) = q_1 \Delta f$$

This can more conveniently be written as

$$i\,q_0(ap_1 - q_1) + a\nabla_\xi p_1 \cdot \nabla_x q_1 - q_1 \nabla_\xi p_1 \cdot \nabla_x a = q_1 \Delta f$$

This looks rather complicated but we can simplify it by noticing that we are free to choose $\nabla f = 0$, because we are only interested in local properties. In addition we can take $\partial^2 f/\partial x_i \partial x_j = 0$ if $i \neq j$. Then noting that p_1 and q_1 satisfy the equations

$$\begin{aligned} q_1^2 + 2\nabla f \cdot \tilde{\xi}\, q_1 + a(\left|\tilde{\xi}\right|^2 - \tau^2) = 0 \\ a p_1^2 + 2\nabla f \cdot \tilde{\xi}\, p_1 + \left|\tilde{\xi}\right|^2 - \tau^2 = 0 \end{aligned}$$

we can easily calculate that

$$\begin{aligned} \frac{\partial p_1}{\partial \xi_i} &= -\frac{\xi_i}{p_1} \\ \frac{\partial q_1}{\partial x_i} &= -\frac{\partial^2 f}{\partial x_i^2}\xi_i \end{aligned}$$

where $i > 1$. If $i = 1$ then $\partial q_1/\partial x_1 = 0$. This leads to

$$q_0 = \frac{i}{2}\left(\Delta f - \frac{1}{\tau^2 - \left|\tilde{\xi}\right|^2}\sum_{i=2}^{n}\frac{\partial^2 f}{\partial x_i^2}\xi_i^2\right)$$

where evidently the partial derivatives of f are evaluated at the point where $\nabla f = 0$. Let us interpret the condition obtained. First recall that if $\bar{\beta} : \mathbb{R} \to \mathbb{R}^2$ is a differentiable curve, $\bar{\beta}(s) = (x(s), y(s))$, then its curvature $\bar{\kappa}$ is defined as follows

$$\bar{\kappa} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$$

Now in our case we have a differentiable surface (the boundary of Ω) given by mapping $\beta(\tilde{x}) = (f(\tilde{x}), \tilde{x})$. Then the intersections of this surface with the coordinate planes $A_j = \{\, x \in \mathbb{R}^n \,|\, x_i = 0 \; i > 1, \; i \neq j \,\}$ can be considered as curves, and identifying A_j and $\mathbb{R}^2$ we can calculate the corresponding curvature with the above formula. This gives immediately that $\kappa_j = \partial^2 f/\partial x_j^2$. Let us remark that there is no intrinsic way to define the sign of the curvature, but in our case we take it to be positive in convex parts of the boundary and negative otherwise (more precisely, $\kappa_j > 0$ if the corresponding curve in A_j is convex). Note that the individual κ_j obviously depend on the choice of the coordinate system, but their sum does not (see [SP], the invariant character of this sum is also a consequence of the fact that Δ does not depend on the coordinate system). Collecting all this together we can state the following

Theorem 4 *Suppose that $\Omega \subset \mathbb{R}^n$ is an open set with smooth boundary Γ. Then the first two terms of the asymptotic expansion of the symbol of the absorbing boundary operator for the equation (2.2) are given by*

$$q_1(\xi, \tau) = -\sqrt{\tau^2 - \left|\tilde{\xi}\right|^2}$$

$$q_0(\xi,\tau) = \frac{i}{2}\left(\sum_{j=2}^{n}\kappa_j - \frac{1}{\tau^2 - \left|\tilde{\xi}\right|^2}\sum_{j=2}^{n}\kappa_j\xi_j^2\right)$$

Then we approximate the square root by rational functions as before and drop the second sum in q_0. This dropping can be justified by noticing that if the direction of the propagation is almost normal to the boundary then this term is 'small'. Of course the other reason is simply that the corresponding differential operator would be very complicated. Anyway we then get the following boundary conditions.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial n} + \frac{u}{2}\sum_{j=2}^{n}\kappa_j = 0$$

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t \partial n} + \frac{1}{2}\frac{\partial u}{\partial t}\sum_{j=2}^{n}\kappa_j - \frac{1}{2}\tilde{\Delta} u = 0$$

Next we would like to treat the variable coefficient case. Recall that the (local) wave speed is then $\sqrt{\mu(x)/\rho(x)}$. Taking the point of view that this variation is slow we can treat it as 'locally constant'. This is called freezing the coefficients and with this assumption we get the new boundary conditions simply by scaling appropriately the above conditions which gives

$$\sqrt{\rho(x)\mu(x)}\frac{\partial u}{\partial t} + \mu(x)\frac{\partial u}{\partial n} + \frac{\mu(x)u}{2}\sum_{j=2}^{n}\kappa_j = 0 \tag{2.8}$$

$$\sqrt{\rho(x)\mu(x)}\frac{\partial^2 u}{\partial t^2} + \mu(x)\frac{\partial^2 u}{\partial t \partial n} + \frac{\mu(x)}{2}\frac{\partial u}{\partial t}\sum_{j=2}^{n}\kappa_j - \sqrt{\rho(x)\mu(x)}\frac{\mu(x)}{2\rho(x)}\tilde{\Delta} u = 0$$

In [EM2] it is reported that numerically this is reasonable: the quality of the absorbing conditions is hardly worse when the coefficients are freezed. The quantity $\sqrt{\rho(x)\mu(x)}$ is sometimes called the wave impedance. Then we should show that these boundary conditions yield well-posed problems which will be done in the next section.

Before that, however, we recall another approach to absorbing boundary conditions. Consider an 'obstacle' $\Omega \subset \mathbb{R}^n$ where Ω is compact and connected. A signal is sent towards Ω which scatters it. Under suitable hypothesis one can show that there is an asymptotic (far field) expansion of the scattered signal of the form

$$u_s = \sum \frac{f_j}{r^j}$$

where r is the radius vector and f_j does not depend on r. Then one can construct differential operators which annihilate successive terms of the expansion and take these operators to be the absorbing boundary conditions. In spirit this similar to approximating $\sqrt{\tau - \left|\tilde{\xi}\right|^2}$ by higher order rational functions. Also there is a correspondence between 'far from the obstacle' and 'normal incidence'. We will not pursue this further but refer to [BT].

3 Variational Formulation

Our next task is then to show that the boundary conditions (2.8) lead to well-posed problems. For the first boundary condition one can use variational formulation in an appropriate Sobolev space, but for the second one this is not possible. The problem is that we have the Laplace-Beltrami operator on the boundary and a priori the solution is not regular enough for this to make sense. In the next section we will then consider the second condition from another point of view.

3.1 Abstract Results

We will follow [DL8] in the formulation of our problem and then modify some proofs to suit our purposes. We take all our spaces to be real to simplify the notation. Let V and H be two Hilbert spaces, $V \subset H$ with continuous injection. We denote the norm of V (resp. H) by $\|\cdot\|$ (resp. $|\cdot|$). Let b and $d : V \times V \to \mathbb{R}$ and $a : H \times H \to \mathbb{R}$ be continuous bilinear forms. We suppose that the following conditions are satisfied.

(C1)	$b(u,v) = b(v,u) \qquad \forall\, u,v \in V$	
(C2)	$\exists\, \lambda, \alpha > 0$ such that $b(u,u) \geq \alpha\|u\|^2 - \lambda|u|^2$	$\forall\, u \in V$
(C3)	$d(u,u) \geq 0 \qquad \forall\, u \in V$	
(C4)	$a(u,u) \geq a_*|u|^2 \qquad \forall\, u \in H$	

To these forms we can associate the continuous operators B and $D : V \to V^*$ (V^* being the dual of V) and $A : H \to H$ as follows

$$\begin{aligned} (Au, v) &= a(u,v) \\ \langle Bu, v \rangle &= b(u,v) \\ \langle Du, v \rangle &= d(u,v) \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ means the duality pairing between V and V^* and $(\cdot,\cdot)$ inner product in H. Let $I = [0,T]$ and denote by $L^2(I,V)$ the space of functions $u : I \to V$ which satisfy $\int_0^T \|u(t)\|^2 dt < \infty$. Similarly $u \in L^\infty(I,V)$ means that $\sup_{t\in I} \|u(t)\| < \infty$ and of course instead of V we can use H or V^* with obvious modifications in the definition. We can now formulate our problem as follows: suppose we are given $u_0 \in V$, $u_1 \in H$ and $f \in L^2(I,H)$; then we want to find u such that

(P1)	$u \in L^2(I,V)$ and $u_t \in L^2(I,H)$
(P2)	$\partial_t a(u_t, v) + \partial_t d(u,v) + b(u,v) = \langle f, v\rangle$
	$\forall\, v \in V$ in the sense of distributions
(P3)	$u(0) = u_0 \in V$ and $u_t(0) = u_1 \in H$

More precisely the left and right hand sides of (P2) are equal in $\mathcal{D}^*(I)$ (the set of distributions on I, similarly the set of test functions on I will be denoted by $\mathcal{D}(I)$). The condition (P1) is a kind of minimal amount of regularity that is necessary. Using the operators A, B and D (P2) can also be written as equality in $\mathcal{D}^*(I, V^*)$ as follows

$$\partial_t A u_t + \partial_t D u + B u = f \tag{3.1}$$

This is because the functions of the form $\psi(t)v$ are dense in $\mathcal{D}(I, V)$ ($\psi \in \mathcal{D}(I)$ and $v \in V$). Then we can state the following

Theorem 5 *The problem formulated above admits a unique solution.*

Proof We will need the classical lemma of Gronwall which we recall for convenience.

Lemma 1 *Suppose that $f \in L^\infty(I)$ and $a \in L^1(I)$, a, $f \geq 0$. If*

$$f(t) \leq \int_0^t a(s) f(s) ds + C$$

for $t \in I$ where C is some positive constant then

$$f(t) \leq C \exp\left(\int_0^t a(s) ds\right)$$

Let us start by proving the uniqueness. The ordinary way to proceed is to replace v by u_t in the variational formulation and then to derive the required energy estimates. However, this cannot be done in the present case because to be able to use it u_t should be in $L^2(I, V)$ while it is only in $L^2(I, H)$. So we use the following trick instead (due to Ladyzhenskaia). Define φ as

$$\varphi(t) = \begin{cases} -\int_t^s u(\sigma) d\sigma & t \leq s \\ 0 & t \geq s \end{cases}$$

Evidently $\varphi \in L^2(I, V)$. Then let us write (3.1) as

$$\partial_t (A u_t + D u) = -B u + f \tag{3.2}$$

The right hand side is certainly in $L^2(I, V^*)$ and since left and right hand sides are equal as distributions the left hand side must also be in $L^2(I, V^*)$. So multiplying both sides by φ is possible. This gives

$$\int_0^T \langle \partial_t (A u_t + D u), \varphi \rangle + \langle B u, \varphi \rangle \, dt = \int_0^T (f, \varphi) \, dt$$

Then integrating by parts and passing to the bilinear forms leads to

$$\int_0^T b(u, \varphi) - a(u_t, \varphi_t) - d(u, \varphi_t) \, dt = \int_0^T \langle f, \varphi \rangle \, dt + a(u_1, \varphi(0)) + d(u_0, \varphi(0))$$

Because the problem is linear it is sufficient to show that $u_0 = u_1 = f = 0$ implies $u = 0$. Putting then $u_0 = u_1 = f = 0$ into the above equation and using the definition of φ gives

$$\begin{aligned} \int_0^s b(u,\varphi) - a(u_t,u) - d(u,u)\,dt &= \\ \int_0^s \partial_t(b(\varphi,\varphi) - a(u,u))/2 - d(u,u)\,dt &= 0 \end{aligned}$$

Then integrating again by parts and using the property (C3) we get

$$a(u(s),u(s)) + b(\varphi(0),\varphi(0)) \leq 0$$

By (C2) and (C4) we then have

$$|u(s)|^2 + \|\varphi(0)\|^2 \leq C|\varphi(0)|^2$$

where C is some suitable constant. This gives immediately

$$|u(s)|^2 \leq C \int_0^s |u(t)|^2 dt$$

Then we can put $s = T$ and conclude by Gronwall's lemma that $u = 0$ and that consequently the solution is unique.

To prove the existence we consider a family of finite dimensional spaces $V_m \subset V$ such that when $m \to \infty$, $V_m \to V$. This means that for any $v \in V$ there exist a sequence $v^m \in V_m$ such that $v^m \to v$. Now let $\{\varphi_j^m\}$ be a basis of V_m. Next we see that by putting

$$u^m = \sum_j g_j^m(t)\varphi_j^m(x)$$

the problem (P1)–(P3) (replacing V by V_m) reduces to a system of (linear) ordinary differential equations. For this kind of system existence and uniqueness are classical results. So all we have to do is to show that when $m \to \infty$ then $u^m \to u$ where u is the solution to our original problem.

Let us then derive the energy estimates for u^m. In the finite dimensional case there is no problem of regularity so we readily get

$$\int_0^T \partial_t a(u_t^m, u_t^m)/2 + b(u^m, u_t^m) + d(u_t^m, u_t^m)\,dt = \int_0^T (f, u_t^m)\,dt$$

Then integrating by parts and remembering (C3) leads to

$$\begin{aligned} &a(u_t^m(T), u_t^m(T)) + b(u^m(T), u^m(T)) \leq \\ &a(u_t^m(0), u_t^m(0)) + b(u^m(0), u^m(0)) + C\int_0^T |f|\,|u_t^m|\,dt \end{aligned}$$

Now we choose $u^m(0)$ and $u_t^m(0)$ in such a way that they converge to u_0 and u_1 respectively so we have the estimates

$$a(u_t^m(0), u_t^m(0)) + b(u^m(0), u^m(0)) \leq C(\|u_0\|^2 + |u_1|^2)$$

Then using (C2) and (C4) with the above inequalities yield

$$|u_t^m(T)|^2 + \|u^m(T)\|^2 \leq C_1 + C_2 \int_0^T |u_t^m|^2 \, dt$$

where C_1 depends on u_0, u_1 and f. Gronwall's lemma then gives immediately that there exists a constant C which does not depend on m such that

$$\begin{aligned} \sup_{t \in I} \|u^m(t)\| &\leq C \\ \sup_{t \in I} |u_t^m(t)| &\leq C \end{aligned}$$

This proves the following

Proposition 1 *The sequence u^m (resp. u_t^m) remains in a bounded set of $L^\infty(I, V)$ (resp. $L^\infty(I, H)$).*

Of course the same is true in $L^2(I, V)$ (resp. $L^2(I, H)$) because the interval I is finite. Now it is well-known that bounded sets in these spaces are weakly (or weakly $*$) relatively compact, so there is a subsequence of u^m and of u_t^m which converge in these weak topologies. However, since we have already shown that the solution is unique, it must be the whole sequence which converges. Then recalling that operators A, B and D are continuous in appropriate spaces we have the following results.

Proposition 2

$$\begin{array}{rll} u^m \to u & \text{in } L^\infty(I, V) \text{ weak} * & \text{and in } L^2(I, V) \text{ weak} \\ u_t^m \to u_t & \text{in } L^\infty(I, H) \text{ weak} * & \text{and in } L^2(I, H) \text{ weak} \\ Au^m \to Au & \text{in } L^2(I, H) \text{ weak} & \\ Bu^m \to Bu & \text{in } L^2(I, V^*) \text{ weak} & \\ Du^m \to Du & \text{in } L^2(I, V^*) \text{ weak} & \end{array}$$

Then the next step is to show that the function u found above verifies the condition (P2). Take a sequence v^m such that $v^m \to v$ strongly in V. Then define $\psi^m = \varphi v^m$ where $\varphi \in \mathcal{D}(I)$ so that $\psi^m \to \psi$ strongly in $L^2(I, V)$. Now we can write (P2) as

$$\begin{aligned} &\int_0^T \langle \partial_t(Au_t^m + Du^m), \psi^m \rangle + \langle Bu^m, \psi^m \rangle \, dt = \\ &\int_0^T -(\langle Au_t^m, v^m \rangle + \langle Du^m, v^m \rangle)\varphi_t + \langle Bu^m, v^m \rangle \varphi \, dt = \\ &= \int_0^T (f, \psi^m) \, dt \end{aligned}$$

Now using the above proposition and the fact that if $x^m \to x$ in V strongly and if $y^m \to y$ in V^* weakly then $\langle x^m, y^m \rangle \to \langle x, y \rangle$ we get

$$\int_0^T -(\langle Au_t, v\rangle + \langle Du, v\rangle)\varphi_t + \langle Bu, v\rangle\varphi\, dt =$$
$$= \int_0^T (f, v)\varphi\, dt$$

which is the condition (P2). To conclude we have still to take care of the initial conditions. To this end take ψ^m as above except that now $\varphi(0) \neq 0$ (and still $\varphi(t) = 0$ near T). Proceeding as before and passing to the limit we get

$$\int_0^T -(\langle Au_t, v\rangle + \langle Du, v\rangle)\varphi_t + \langle Bu, v\rangle\varphi\, dt =$$
$$= a(u_1, v)\varphi(0) + \int_0^T (f, v)\varphi\, dt$$

On the other hand starting directly from u we get the same expression except that instead of u_1 we have $u_t(0)$. So we get

$$(Au_1, v) = (Au_t(0), v) \qquad \text{for all } v \in V$$

From the property (C4) it follows that A is injective; this combined with the denseness of V in H shows that $u_1 = u_t(0)$. Next taking the same ψ^m as above we have

$$\int_0^T (u_t^m, \psi^m)\, dt = -(u^m(0), \psi(0)) - \int_0^T (u^m, \psi_t^m)\, dt$$

Passing to the limit we get

$$\int_0^T (u_t, \psi)\, dt = -(u_0, \psi(0)) - \int_0^T (u, \psi_t)\, dt$$

Starting directly from u we get the same thing except that instead of u_0 we have $u(0)$ so that

$$(u_0, v) = (u(0), v) \qquad \text{for all } v \in V$$

which shows that $u_0 = u(0)$. So the proof of the theorem is finally complete.

■

Let us remark that working a little harder one could prove that the solution $u : I \to X$ is continuous where X is some appropriate space (see [LM]). However, the above result is sufficient in the present context.

3.2 First Order Conditions

Now consider the equation (2.5) defined in the domain $\Omega \subset \mathbb{R}^n$ where is Ω an open and bounded set which is locally on one side of its Lipschitz continuous boundary Γ. Multiplying by a test function and using the Green's formula gives

$$\partial_t(\rho(x)u_t, v) + \int_\Omega \mu(x)\nabla u \cdot \nabla v - \int_\Gamma v\, \mu(x)\nabla u \cdot n = (f, v)$$

where n is the exterior unit normal. We consider a simple 'obstacle problem'; that is the boundary consists of two disjoint homeomorphic images of the unit sphere Γ_1 and Γ_2 (two disjoint Jordan curves when $n = 2$), and on Γ_1 we take the first order absorbing boundary condition (the first condition in (2.8)) and on Γ_2 the homogeneous Dirichlet condition. This gives

$$\begin{aligned} \partial_t(\rho(x)u_t, v) + \int_\Omega \mu(x)\nabla u \cdot \nabla v + & \\ \partial_t \int_{\Gamma_1} \sqrt{\rho(x)\mu(x)}uv + \frac{1}{2}\int_{\Gamma_1} \mu(x)uv\sum_{j=2}^{n} \kappa_j &= (f, v) \end{aligned} \tag{3.3}$$

The three bilinear forms are then

$$\begin{aligned} a(u, v) &= (\rho(x)u, v) \\ b(u, v) &= \int_\Omega \mu(x)\nabla u \cdot \nabla v + \frac{1}{2}\int_{\Gamma_1} \mu(x)uv\sum_{j=2}^{n} \kappa_j \\ d(u, v) &= \int_{\Gamma_1} \sqrt{\rho(x)\mu(x)}uv \end{aligned}$$

The spaces V and H are then simply the usual $V = \{\, u \in H^1(\Omega) \,|\, u = 0 \text{ on } \Gamma_2 \,\}$ and $H = L^2(\Omega)$. Let us then suppose that $\rho \in L^\infty(\Omega)$, $\mu \in L^\infty(\Omega)$ and $\kappa_j \in L^\infty(\Gamma)$. The condition on κ_j is satisfied if the boundary is piecewise smooth and Lipschitz. With these hypothesis we obtain

Theorem 6 *The bilinear forms a, b and d are continuous and the conditions (C1) - (C4) are satisfied.*

Proof As is well known the functions in $H^1(\Omega)$ have a trace on the boundary and we have the estimate $\|u\|_{L^2(\Gamma)} \le C\,\|u\|$ for some constant C. This shows that the bilinear forms are continuous.
Evidently the property (C1) holds. Then taking $\rho(x) \ge \rho_* > 0$ and $\mu(x) \ge 0$ clearly imply (C3) and (C4). To get (C2) we have to require a little more: $\mu(x) \ge \mu_* > 0$. This turns out to be sufficient if we recall the following result (see [DL4]): for every $\varepsilon > 0$ there corresponds $M(\varepsilon)$ such that

$$\|u\|^2_{L^2(\Gamma)} \le \varepsilon\|u\|^2 + M(\varepsilon)|u|^2$$

Let us denote by C the following quantity

$$\frac{\|\mu(x)\|_{L^\infty(\Gamma)}}{2}\sum_{j=2}^{n}\|\kappa_j\|_{L^\infty(\Gamma)}$$

Then picking $\varepsilon = \mu_*/2C$ we see that (C2) holds with $\alpha = \mu_*/2$ and $\lambda = CM(\varepsilon)$.

■

Then evidently we get

Corollary 1 *The problem (3.3) is well posed.*

4 Stability of Hyperbolic Systems

4.1 Theory of Kreiss

We start by describing the theory of Kreiss (see [KR]), and then see if it can be applied to the present situation, namely to prove that the second boundary condition of (2.8) also yield a well posed problem. Consider the following first order system.

$$u_t - \sum_{j=1}^{n} A_j u_{x_j} - Cu = f \tag{4.1}$$

The matrices A_j and C need not be constant. Since the whole analysis is local, however, we might as well suppose that after some preliminary transformations our domain is the half space $\mathbb{R}^n_+$ and A_j are constant. The zeroth order term Cu has no effect on the stability questions so we can take $C = 0$. Next recall the following

Definition 3 *The system (4.1) is called (strictly) hyperbolic if the eigenvalues of $\sum A_j\xi_j$ are real (and distinct), where ξ_j are real and $|\xi| = 1$.*

Supposing then (4.1) to be hyperbolic one can simplify further (but without loss of generality) and take A_1 to be

$$A_1 = \begin{pmatrix} A_1^I & 0 \\ 0 & a_1^{II} \end{pmatrix}$$

where A_1^I (resp. A_1^{II}) is negative (resp. positive) and both are diagonal. We partition u into u_I and u_{II} with dimensions l_1 and l_2 corresponding to matrices A_1^I and A_1^{II}. It is then evident that there are l_1 'right going' characteristics (that is characteristics that do not hit the boundary) and consequently we can then impose only l_1 boundary conditions which we shall write as

$$u_I = Su_{II} + g$$

Then taking the Fourier transform in space (except in x_1 direction) and Laplace transform in time we obtain the following system of ODEs.

$$\frac{d\hat{u}}{dx_1} = M(s,\tilde{\xi})\hat{u} + \hat{F}$$

where s is the Laplace variable and $\tilde{\xi} = \begin{pmatrix} \xi_2 & \dots & \xi_n \end{pmatrix}$ the Fourier variable and M is given by

$$M = A_1^{-1}\left(sI - i\sum_{j=2}^{m} A_j\xi_j\right)$$

Now consider the homogeneous or eigenvalue problem

$$\begin{aligned} \frac{d\hat{u}}{dx_1} &= M(s,\tilde{\xi})\hat{u} \\ \hat{u}_I &= S\hat{u}_{II} \end{aligned} \tag{4.2}$$

We can interpret this as an initial value problem where the initial values are constrained to lie in the $l_2 = n - l_1$ dimensional subspace. For $\hat{u}$ to be an eigenfunction we require it to belong to $L^2(\mathbb{R}_+)$ and the corresponding s is called the eigenvalue. Note that $\Re s > 0$ by definition of the Laplace transform. However, the initial value problem (4.2) can sometimes have bounded solutions with $\Re s = 0$. If this is the case we call such an s the generalized eigenvalue. Now by definition we say that (4.2) or (4.1) satisfies the *Kreiss condition* if the problem (4.2) does not admit eigenvalues or generalized eigenvalues. Now we can state the main theorem of the Kreiss theory.

Theorem 7 *If the problem (4.1) satisfies the Kreiss condition, then it is stable, that is we have the estimate*

$$\int_0^T \|u(0,\tilde{x},t)\|^2_{\mathbb{R}^{n-1}} + \int_0^T \|u(x,t)\|^2_{\mathbb{R}^n_+} \le C\Big(\int_0^T \|g(\tilde{x},t)\|^2_{\mathbb{R}^{n-1}} + \int_0^T \|F(x,t)\|^2_{\mathbb{R}^n_+}\Big)$$

Here evidently $\tilde{x} = \begin{pmatrix} x_2 & \dots & x_n \end{pmatrix}$. We remark that it is possible also to treat the case $T = \infty$, but then one has to use weighted norms instead.

Now let us see what happens when we write the wave equation in $\mathbb{R}^2_+$ with second order boundary conditions as a first order system. Putting $v_1 = u_{yy}$, $v_2 = u_{yt}$, $v_3 = u_{tt}$ and $v_4 = u_{xt}$ we get

$$\begin{aligned} v_t - \tilde{A}v_x - \tilde{B}v_y &= 0 \\ \tilde{S}\cdot v = 0 & \end{aligned}$$

where

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \tilde{S} = \begin{pmatrix} -1/2 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

Transforming $\tilde{A}$ to the diagonal form we get

$$\begin{aligned} w_t - Aw_x - Bw_y &= 0 \\ S \cdot w = 0 \end{aligned}$$

where $w = P^{-1}v$ and

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix} \qquad S = \begin{pmatrix} -1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

So A does not have full rank and the theory of Kreiss as such does not apply.

4.2 Characteristic Boundaries

The theory of Kreiss has been generalized by Majda and Osher (see [MO]) to the case of uniformly characteristic boundary. In case of the half space we call the boundary *characteristic* if A_1 in (4.1) does not have full rank and *uniformly characteristic* if the rank of A_1 remains constant in some neighborhood of the boundary (obviously these definitions generalize easily to any boundary). Now the aim is to get same kind of estimates as in the preceding theorem. To this end we take A_1 in (4.1) to be of the form

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_1^I & 0 \\ 0 & 0 & A_1^{II} \end{pmatrix}$$

where A_1^I and A_1^{II} are as before. We partition u as $u = \begin{pmatrix} u_0 & u_* \end{pmatrix} = \begin{pmatrix} u_0 & u_I & u_{II} \end{pmatrix}$. Doing the same Laplace Fourier transform as before, we obtain l_0 algebraic equations and a system of $l_1 + l_2$ differential equations. We can eliminate the u_0 components from the differential system which can then be written as

$$\frac{d\hat{u}_*}{dx_1} = M(s, \tilde{\xi})\hat{u}_*$$

However, there is no guaranty that we can eliminate u_0 from the boundary conditions. Anyway we shall make the following hypothesis.

Assumption 1 All A_j are symmetric.

Assumption 2 Consider the matrix B defined by

$$B = \sum_{j=2}^{n} A_j \xi_j = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix}$$

where B_{11} is $l_0 \times l_0$ matrix. We require that the eigenvalues of B_{11} are real and distinct for all real ξ with $|\xi| = 1$.
Assumption 3 The boundary conditions can be written as

$$\hat{u}_I = S\hat{u}_{II}$$

With these supplementary hypothesis we get the 'same' theorem as before.

Theorem 8 *Kreiss condition implies stability.*

Then comparing to our example we see that none of the assumptions are satisfied....Of course it may be possible to find a transformation such that one of them holds, but it seems impossible to satisfy all of them. Anyway interpreting the wave equation as a first order system introduces zero modes which seem to be completely artificial and it is them that cause all the trouble. We might conclude that one had better to analyse the wave equation directly.

4.3 Wave Equation

The problem with this approach is of course that we lose generality: the wave equation can be an interesting object, but one would not like to build a whole theory for a single equation. However, we shall outline how one can finally prove stability in this case (for more details see [JO2], and for some recent results [HJ]). More complete analysis from slightly different point of view can be found in [HT].
Let us consider the following model problem.

$$\begin{aligned} u_{tt} - \Delta u &= 0 \qquad \text{in } \mathbb{R}^n_+ \\ \mathcal{B}u &= g \qquad \text{in } \mathbb{R}^{n-1} \end{aligned} \tag{4.3}$$

where as usual the boundary is identified with $\mathbb{R}^{n-1}$ and $\mathcal{B}$ is some differential operator to be specified. This form can be used in the constant coefficient case without loss of generality: the problem being linear one can reduce a more general equation to this form by subtracting a suitable solution from the initial problem.
Next, let us suppose that after the usual Laplace-Fourier transform the boundary condition can be written as

$$p_1(\tilde{\xi}, s)\frac{\partial \hat{u}}{\partial x_1} + p_2(\tilde{\xi}, s) = \hat{g}(\tilde{\xi}, s)$$

where p_1 (resp. p_2) is homogeneous of degree $m-1$ (resp. m). Elementary calculations show that any bounded solution is of the form

$$\hat{u}(x_1,\tilde{\xi},s) = \varphi(\tilde{\xi},s)\exp(-x_1\sqrt{|\tilde{\xi}|^2 + s^2})$$

where φ is determined by the boundary condition

$$-p_1\varphi\sqrt{|\tilde{\xi}|^2 + s^2} + p_2 = \hat{g}$$

Note that the left hand side of the above equation (denoted by F) is homogeneous of degree m. With these conventions the problem (4.3) is said to satisfy the Kreiss condition if F does not have any zeroes with $\tilde{\xi} \in \mathbb{R}^{n-1}$ and $\Re s \geq 0$ (except of course the trivial solution $s = 0$ and $\tilde{\xi} = 0$). This gives then as before

Theorem 9 *Kreiss condition implies stability, that is we have*

$$\int_0^T e^{-\eta t}\left(\|u_t(x,t)\|^2_{\mathbb{R}^n_+} + \|\nabla u(x,t)\|^2_{\mathbb{R}^n_+}\right) \leq C(\eta)\int_0^T \|g(\tilde{x},t)\|^2_{H^{-m}(\mathbb{R}^{n-1})}$$

Here η is some suitable positive constant. Note that here we get the bound for the energy norm and not for the L^2-norm. Let us then verify that

Corollary 2 *Second order absorbing boundary conditions yield a well posed problem.*

Proof In case of our model problem (4.3) the second order condition is

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial t \partial x_1} - \frac{1}{2}\tilde{\Delta} u = g$$

so that

$$\begin{aligned}
p_1 &= -s \\
p_2 &= s^2 + |\tilde{\xi}|^2/2 \\
F &= s\sqrt{|\tilde{\xi}|^2 + s^2} + s^2 + |\tilde{\xi}|^2/2
\end{aligned}$$

Case 1 Take $s = i\omega$ where $\omega \in \mathbb{R}$. If $|\omega| \leq |\tilde{\xi}|$ then

$$F = i\omega\sqrt{|\tilde{\xi}|^2 - \omega^2} - \omega^2 + |\tilde{\xi}|^2/2$$

which evidently is non zero. Then if $|\omega| > |\tilde{\xi}|$ we obtain

$$F = -\omega\sqrt{|\tilde{\xi}|^2 - \omega^2} - \omega^2 + |\tilde{\xi}|^2/2 < 0$$

Case 2 Taking $\Re s > 0$ we note that $|\tilde{\xi}|^2 + s^2$ can never be real and negative so that by cutting off the negative part of the real line in the complex plane we can uniquely define $\gamma = \sqrt{|\tilde{\xi}|^2 + s^2}$. Then we get immediately

$$\begin{aligned} F &= s\gamma + s^2 + |\tilde{\xi}|^2/2 \\ &= s\gamma + s^2 + (\gamma^2 - s^2)/2 = \frac{1}{2}(s+\gamma)^2 \neq 0 \end{aligned}$$

because $\Re s > 0$ and $\Re \gamma > 0$.

∎

Finally recall that there exist stable absorbing boundary conditions of arbitrary order, see for instance [EM1], [HT] and [BT]. In principle their analysis is similar as developed above, various calculations being of course more cumbersome.

5 Numerical Applications

5.1 Setting Up the Problem

Now of course absorbing boundary conditions have been around already for some time, see for instance [EM1], [EM2], [HT] and [BT], and the conclusion seems to be that the first order conditions are not sufficiently transparent, so that in practical calculations one should use (at least) second order conditions. That curvature should also be taken into account is no secret either, but usually it is simply ignored or it is said that its effect is 'small'. Evidently in the finite difference context where rectangular domains are used the curvature is zero, and anyway the theoretical analysis is usually done in terms of the half space, so that the curvature is easy to forget.

On the other hand there appears the 'corner problem': because differential operators cannot be defined if the boundary is not smooth one has to treat separately the corners. For instance following conditions have been proposed for the right angled corner in two dimensions.

$$\begin{aligned} \sqrt{2}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} &= 0 \\ \frac{3}{2}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} &= 0 \end{aligned}$$

The first can be found in [EM1] and the second in [JO1]. Also different discretizations of the same boundary condition can have different stability properties.

In the variational formulation context the situation is different: because one gets rid of the normal derivative all operators are intrinsically defined either in the domain or on the boundary (considered as a differentiable manifold). However, when triangulating the domain we get necessarily corners on the boundary, so it would seem that we have a corner problem everywhere! Strictly speaking we cannot define neither the curvature nor the Laplace-Beltrami operator on the boundary. Anyway we shall propose a simple way to define discrete curvature and discrete Laplace-Beltrami operator such that the latter is positive definite (and as we shall see this is not automatic).

First let us describe our numerical model. We consider two dimensional wave equation in some domain Ω with zero initial conditions and take the whole boundary Γ to be absorbing. Then we take some suitable source term f which will be specified later. Taking then constant coefficients in the wave equation we obtain the following variational formulation.

$$\int_\Omega u_{tt}v + \int_\Gamma u_t v + \int_\Omega \nabla u \cdot \nabla v + \frac{1}{2}\int_\Gamma \kappa uv + \frac{1}{2}\int_{t=0}^{T}\int_\Gamma \frac{\partial u}{\partial\tau}\frac{\partial v}{\partial\tau} = \int_\Omega fv \qquad (5.1)$$

We use P_1 elements with mass lumping and the usual three point time discretization scheme which takes care of the first term in (5.1). To keep things nicely centered we use $u_t \sim (u^{n+1} - u^{n-1})/2\delta t$ in the second one (where δt is the time step), together with the mass lumped boundary mass matrix. Here the corners do not cause any trouble because no derivatives are involved. Then the third term is just the standard rigidity matrix.

In the fourth term we have the curvature. Let us define the discrete curvature as follows. Let p_1, p_2 and p_3 be three consecutive boundary points (in that order). Then the discrete curvature κ_h at p_2 is taken to be

$$\kappa_h = \frac{4m(K)}{d_{12}d_{23}d_{31}}$$

where $m(K)$ is the area of the triangle whose vertices are p_1, p_2 and p_3 and d_{ij} is the distance between p_i and p_j. Of course it is irrelevant if this triangle is actually in the triangulation, and in fact as a rule it is not because one usually avoids triangles whose vertices are all on the boundary. This formula is reasonable because if the boundary is smooth and p_1 and p_3 tend to p_2 then κ_h converges to the real curvature (see [SP]). Because we have to evaluate integrals we then have to define κ_h on the edges. Taking the simplest possible solution we associate to each edge the mean value of the curvatures of its vertices. So the fourth term in (5.1) can be discretized by using the boundary mass matrix multiplied by (diagonal) curvature matrix.

Then comes finally the fifth term. For the time integration we use the trapetzoidal rule; of course it is not necessary to calculate the whole integral at each time step, instead an intermediate value is stored and then updated. This is completely elementary and one is left with the problem of giving meaning to the boundary integral. More precisely we have to define the tangent vector τ because

$$\frac{\partial u}{\partial\tau}\frac{\partial v}{\partial\tau} = (\nabla u \cdot \tau)(\nabla v \cdot \tau)$$

and ∇u and ∇v are well defined piecewise constant functions. The simplest possibility is to reason by edges: then τ is also piecewise constant and the integration is trivial. This choice might be called 'flat circle' discretization because it is identical to discretizing the Laplacian on an interval of the real line with end points identified. This obviously guarantees that the discrete operator is positive definite but we lose all the information about the geometry of the boundary.

Trying to be a little more sophisticated we recall that the normal (and consequently the tangent) at the vertex is sometimes defined using the mean of the normals of the edges surrounding the vertex. Then calling E the edge whose vertices are p_1 and p_2 with the associated tangent vectors τ_1 and τ_2, it is natural to define τ at any point of E by linear interpolation: if $\tilde{p} = \lambda p_1 + (1-\lambda)p_2$ then $\tilde{\tau} = \lambda\tau_1 + (1-\lambda)\tau_2$ where $0 \le \lambda \le 1$. Then denoting the basis function associated to p_i by φ_i we easily calculate that

$$\begin{aligned}
\int_E \left(\frac{\partial\varphi_1}{\partial\tau}\right)^2 &= l(E)(a_1^2 + a_1a_2 + a_2^2)/3 \\
\int_E \frac{\partial\varphi_1}{\partial\tau}\frac{\partial\varphi_2}{\partial\tau} &= l(E)(2a_1b_1 + a_1b_2 + a_2b_1 + 2a_2b_2)/6
\end{aligned}$$

where $l(E)$ is the length of the edge E and a_i and b_i are given by

$$\begin{aligned}
a_1 &= \nabla\varphi_1 \cdot \tau_1 \\
a_2 &= \nabla\varphi_1 \cdot \tau_2 \\
b_1 &= \nabla\varphi_2 \cdot \tau_1 \\
b_2 &= \nabla\varphi_2 \cdot \tau_2
\end{aligned}$$

However, the resulting difference operator is not positive! It is sufficient to take the case of the right angle, that is

$$\begin{aligned}
\tau_1 &= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\
\tau_2 &= \begin{pmatrix} 1 & 0 \end{pmatrix} \\
\nabla\varphi_1 &= \begin{pmatrix} 1 & 0 \end{pmatrix} \\
\nabla\varphi_2 &= \begin{pmatrix} -1 & -1 \end{pmatrix}
\end{aligned}$$

Then we calculate that

$$2(a_1^2 + a_1a_2 + a_2^2) = \frac{2+3\sqrt{2}}{\sqrt{2}} < |2a_1b_1 + a_1b_2 + a_2b_1 + 2a_2b_2| = \frac{3+4\sqrt{2}}{\sqrt{2}}$$

which shows that the operator cannot be positive definite. Modyfying the above example it is easy to see that one can lose positivity with arbitrarily small angle. Then

one could take the tangent to be $(\tau_1 + \tau_2)/2$, but this does not in general guarantee the positivity either.
Next step might be to try to use curved elements. But that would be totally unsatisfactory because after all the absorbing boundary conditions are used only on the *artificial* boundary so one would like to keep things as simple as possible.
So all in all we shall use the flat circle discretization. Then there is still the right hand side, but its discretization is entirely standard. To write down the fully discrete scheme let us introduce some notations: mass lumped mass matrix in Ω (resp. on Γ) is denoted by M (resp. M_b), rigidity matrix in Ω (resp. on Γ) by R (resp. R_b), the (diagonal) curvature operator by K and the discrete right hand side by F. This gives

$$\begin{aligned}\frac{1}{\delta t^2}M(u^{n+1} - 2u^n + u^{n-1}) + \frac{1}{2\delta t}M_b(u^{n+1} - u^{n-1}) + & \\ Ru^n + \frac{1}{2}KM_bu^n + \frac{1}{4}R_b(u^0 + u^n + 2\sum_{i=1}^{n-1} u^i) &= F^n\end{aligned}$$

In conclusion we might say that the curvature term has two functions: firstly it gives automatically good (or at least reasonable) corner conditions, and secondly it 'restores' some geometrical information that was lost in the flat circle discretization.

5.2 Numerical Results

Next let us consider some examples. Let us take the right hand side to be of the form $f = \delta(x) \otimes g(t)$, where δ is the Dirac measure and g is some (more or less) smooth function. In the first test case we take the domain shown in figure 5.1 where the corner on the left (resp. on the right) is at (-10,0) (resp. at (10,0)) and $h = 10/36 \simeq 0.278$. Then we define g by

$$g(t) = \begin{cases} 100\sin^4(\pi t/L) & 0 \le t \le L \\ 0 & \text{otherwise} \end{cases}$$

and put the Dirac measure at $(3.333, 7.698)$. Note that L cannot really be called the wavelength; we might call it pulsewidth and anyway define $N = L/h$, the number of points per pulsewidth. Then taking $L = 4$ (which gives $N = 14.4$) and $t = 15$ and using $\delta t = 0.2$, we have in figure 5.2 the solution without the curvature term and in figure 5.3 with curvature term. It is clear that in the former case there are rather strong reflections from the corners which are absent in the latter. Then in figure 5.4 there is the unit square with $h = 0.02$. Putting the Dirac measure at $(0.66, 0.78)$, using $\delta t = 0.0125$ and taking the same g as before, but with $L = 0.3$ (and $N = 15$) we have in figure 5.5 the solution at $t = 0.9$ without curvature term and in figure 5.6 with the curvature term. The difference is again obvious. For completeness we show in figures

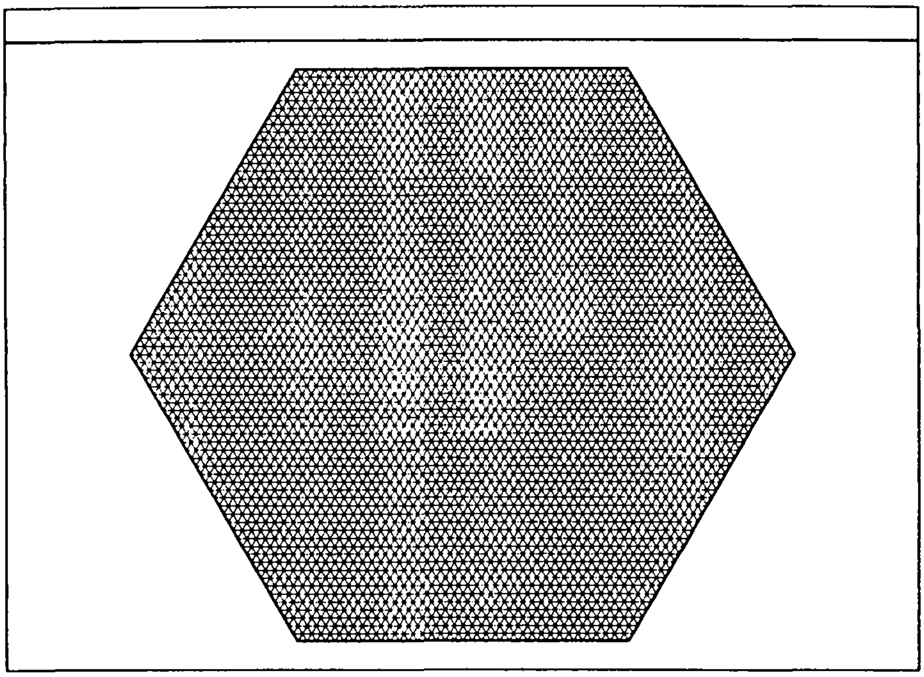

Figure 5.1: First domain and the mesh used.

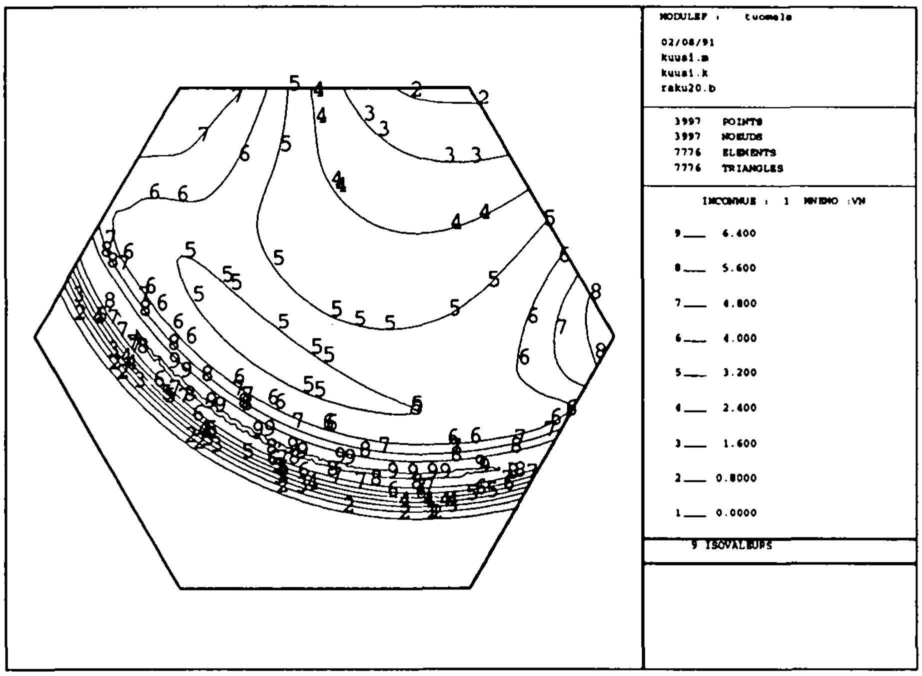

Figure 5.2: Solution at $t = 15$ without curvature, second order condition.

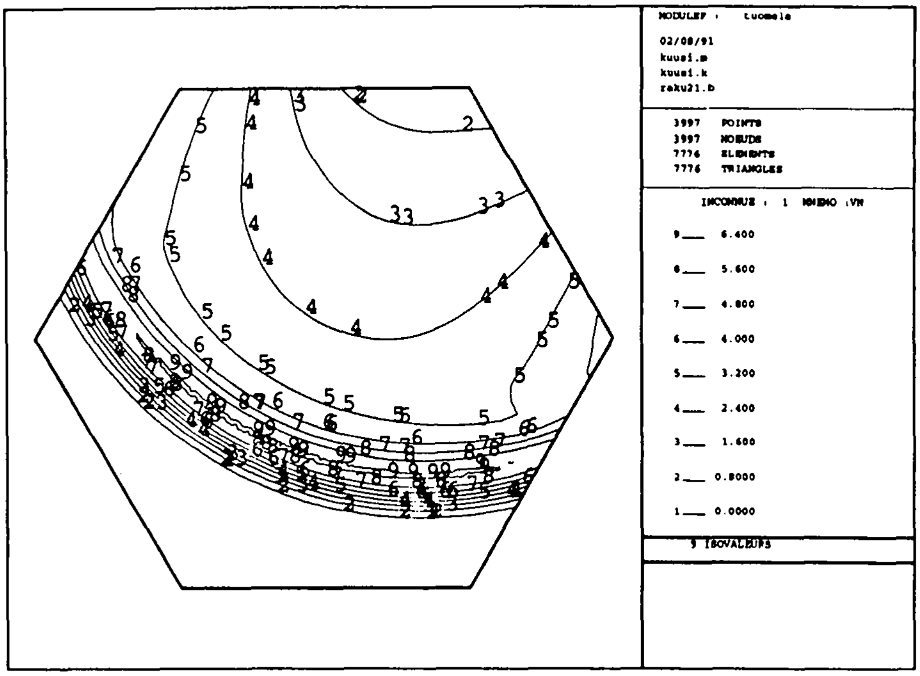

Figure 5.3: Solution at $t = 15$ with curvature, second order condition.

5.7, 5.8, 5.9 and 5.10 the corresponding solutions with first order boundary conditions. As expected, the wave front near the boundary is clearly worse in this case.

The domain of the third test case is shown in figure 5.11, the circle centered at origin with radius 10. The Dirac measure was at $(-6, -4)$, $h = 0.25$, $\delta t = 0.1$, $L = 3$, $N = 12$ and $t = 15$. The time step was so small because there were some small triangles near the boundary; these small triangles also caused some irregularities in the solution near the boundary. The effect of the curvature term can be seen, although it is not so clear as in previous cases.

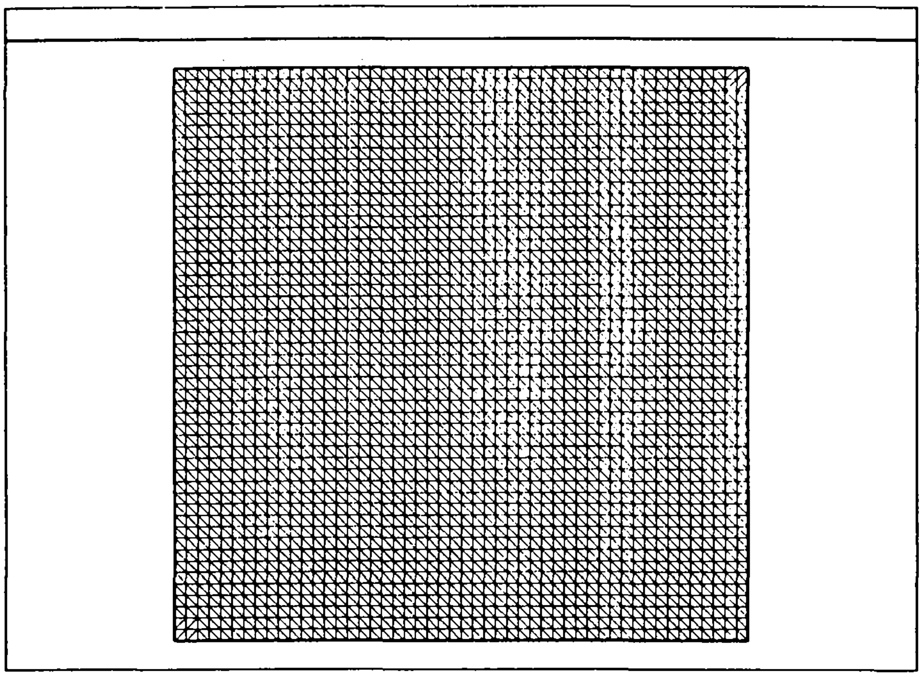

Figure 5.4: Second domain and the mesh used.

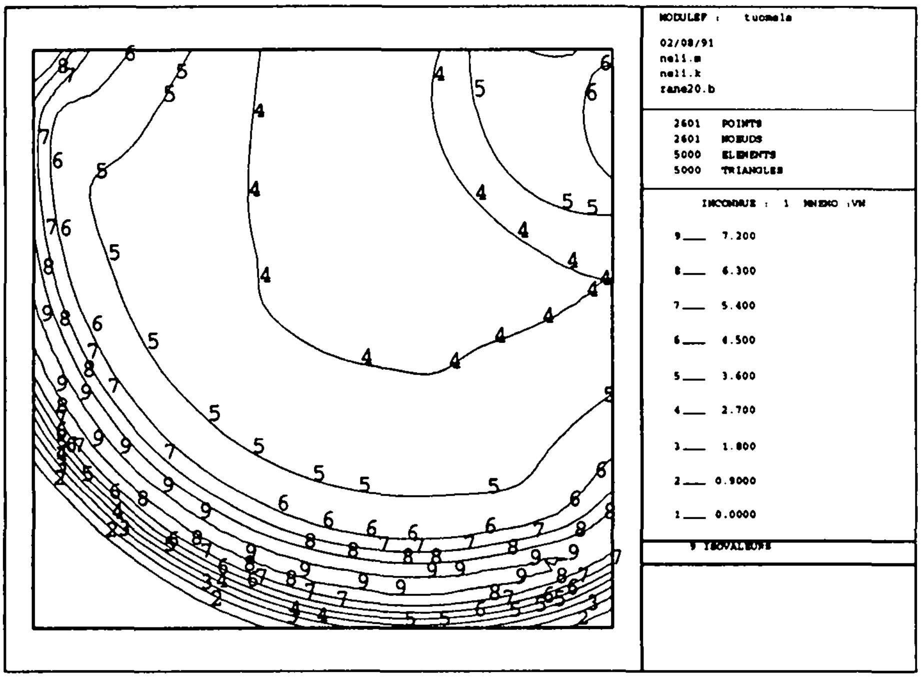

Figure 5.5: Solution at $t = 0.9$ without curvature, second order condition.

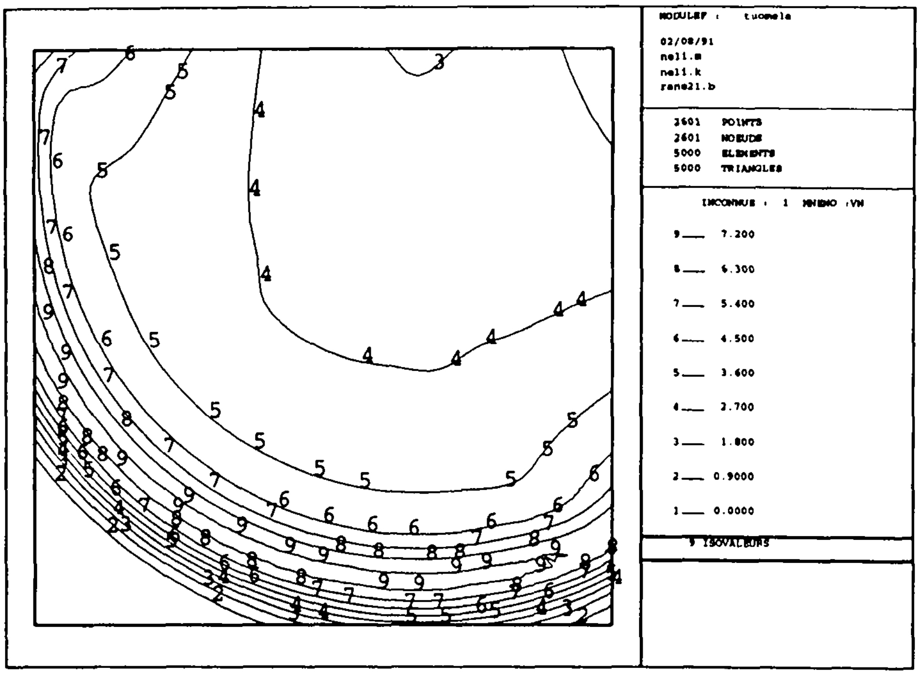

Figure 5.6: Solution at $t = 0.9$ with curvature, second order condition.

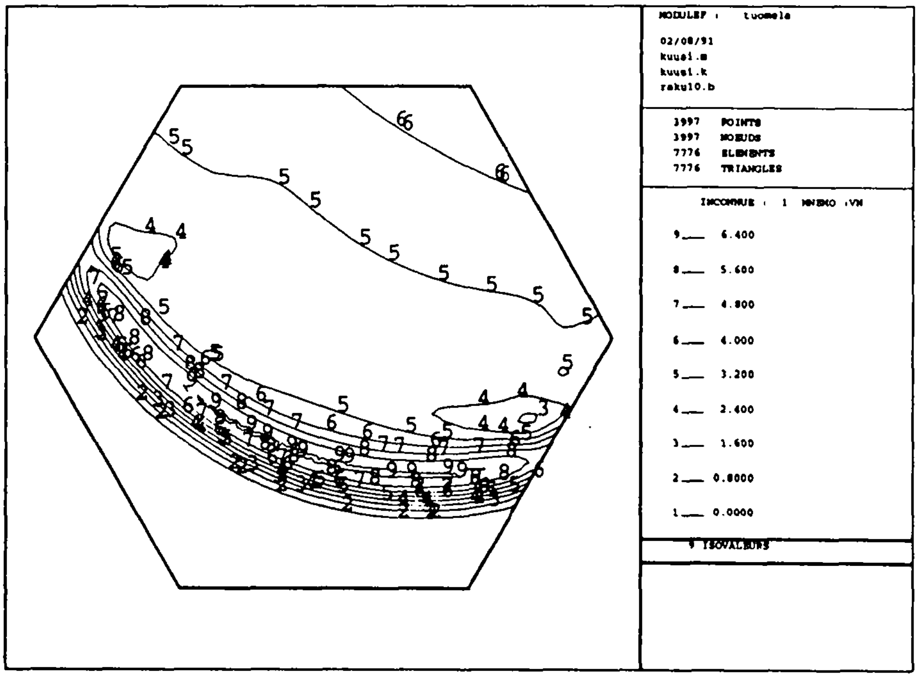

Figure 5.7: Solution at $t = 15$ without curvature, first order condition.

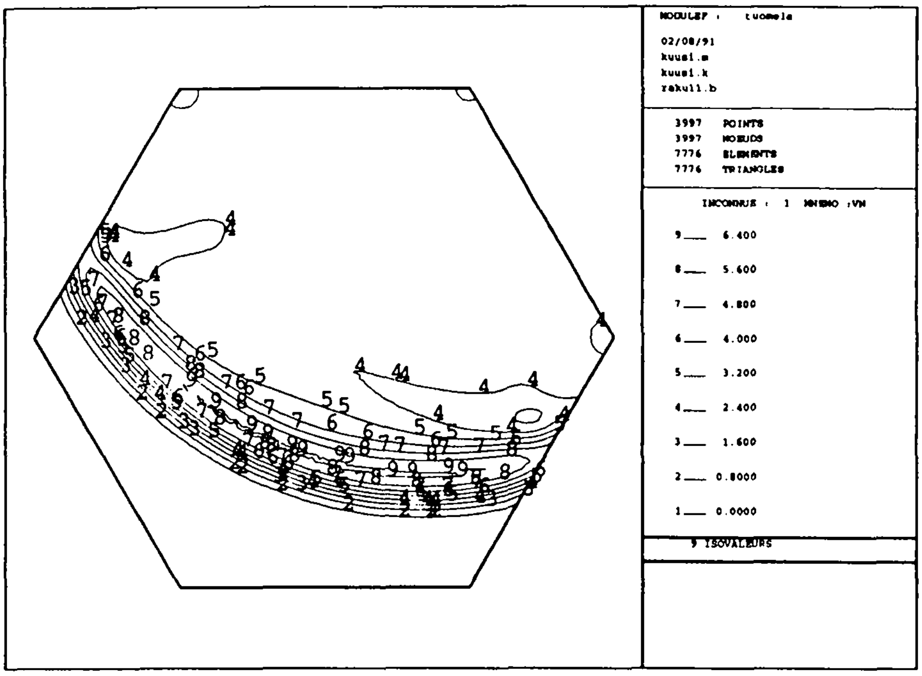

Figure 5.8: Solution at $t = 15$ with curvature, first order condition.

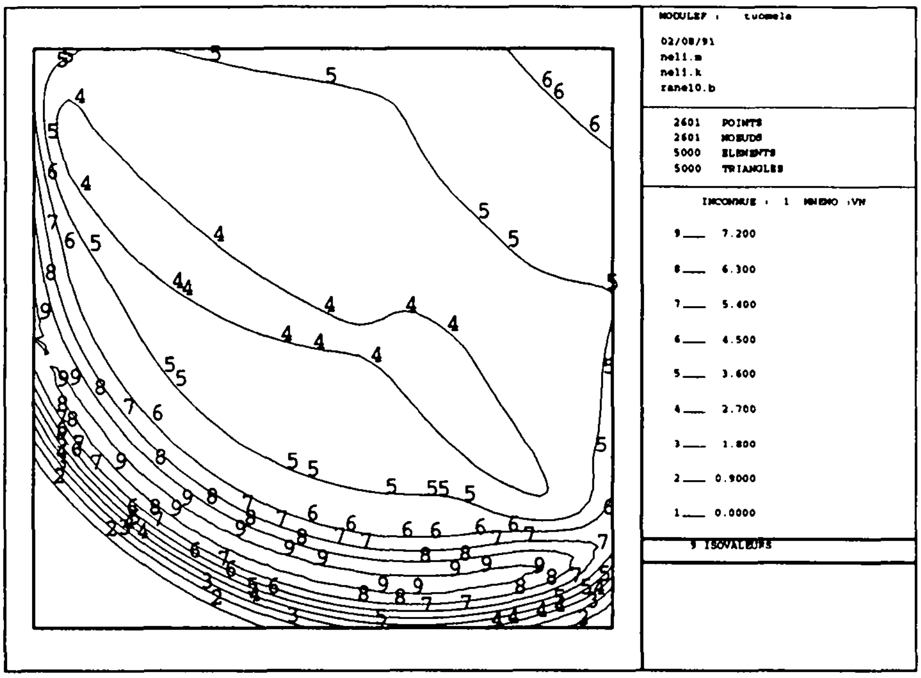

Figure 5.9: Solution at $t = 0.9$ without curvature, first order condition.

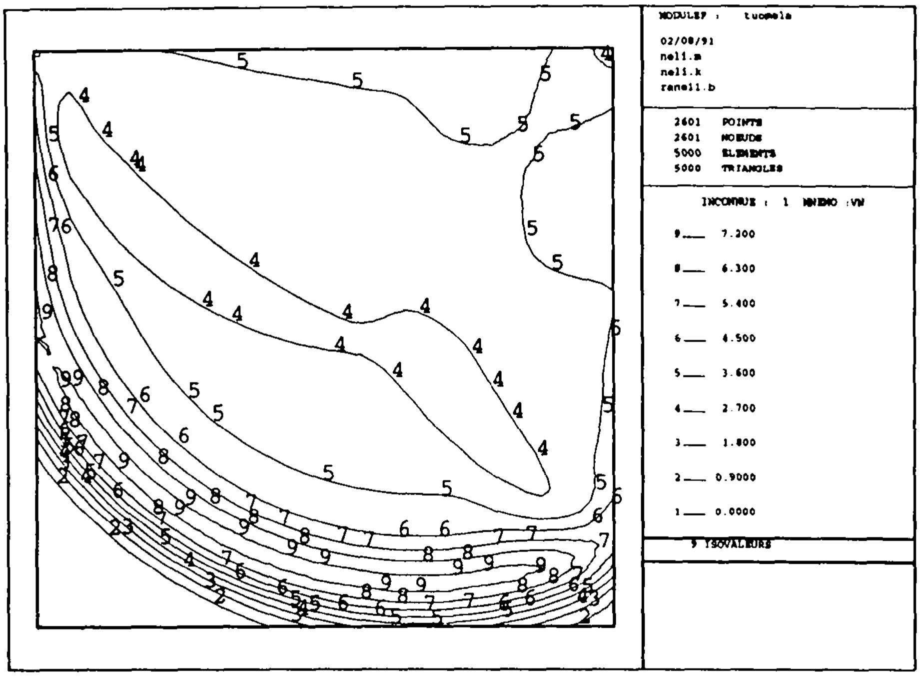

Figure 5.10: Solution at $t = 0.9$ with curvature, first order condition.

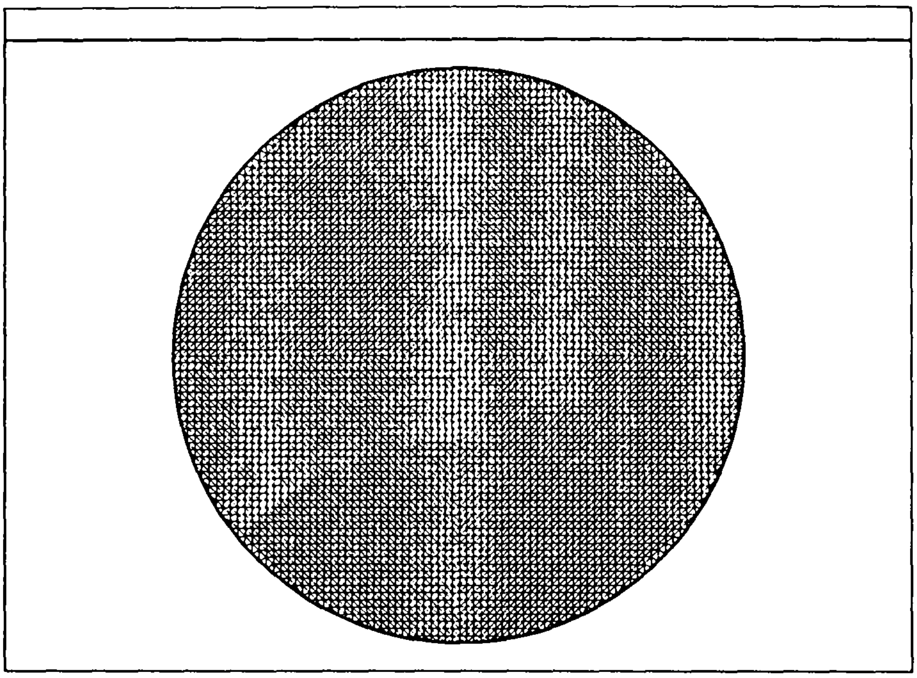

Figure 5.11: Third domain and the mesh used.

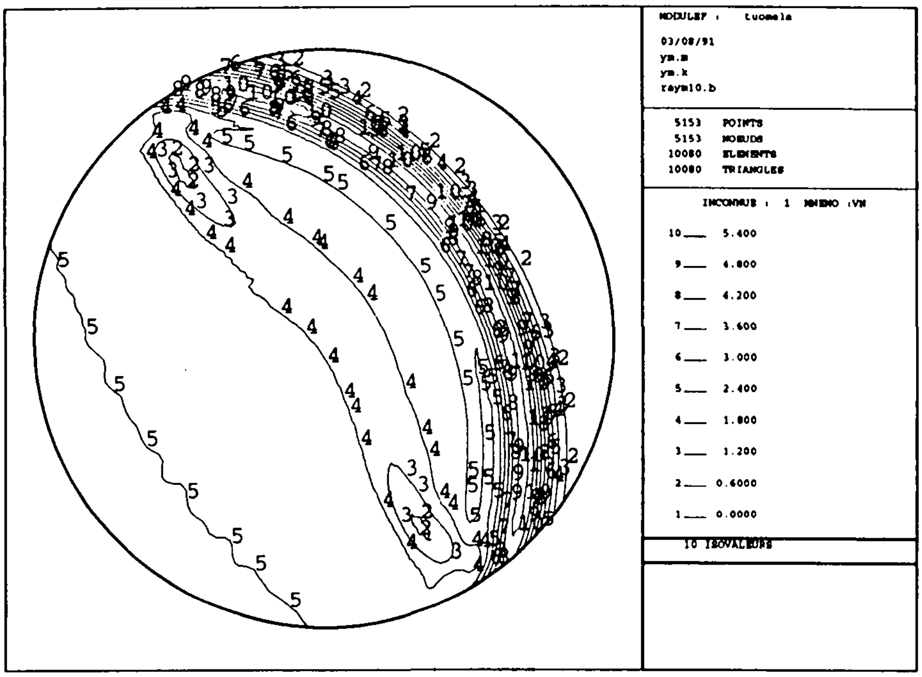

Figure 5.12: Solution at $t = 1.6$ without curvature, first order condition.

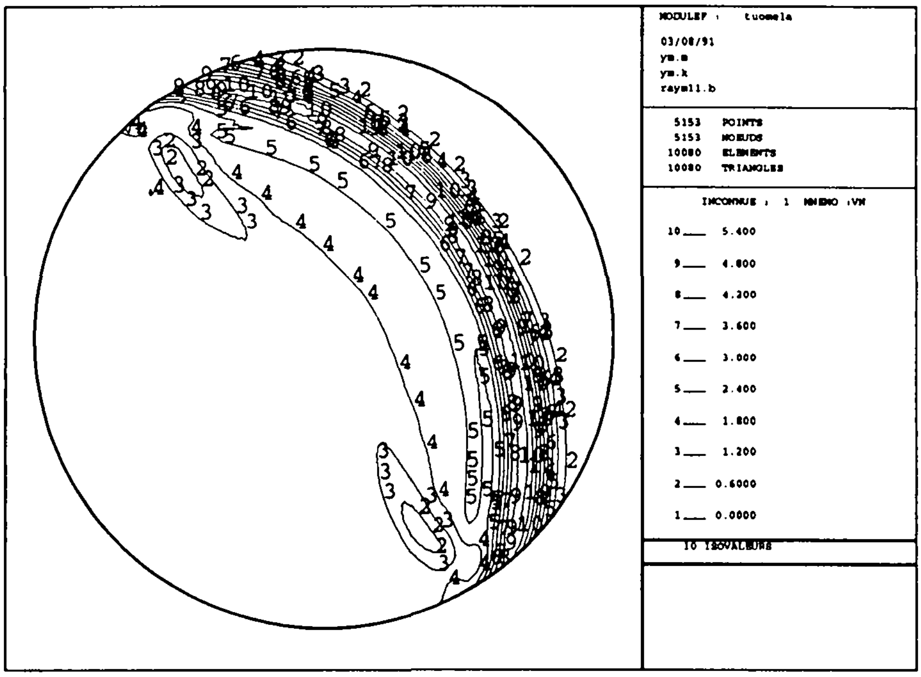

Figure 5.13: Solution at $t = 1.6$ with curvature, first order condition.

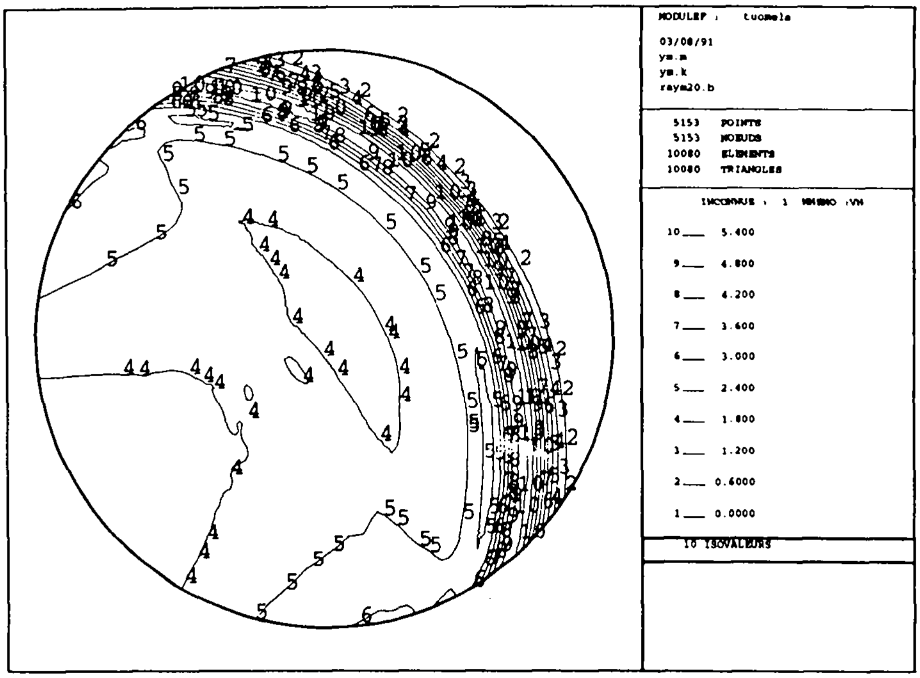

Figure 5.14: Solution at $t = 1.6$ without curvature, second order condition.

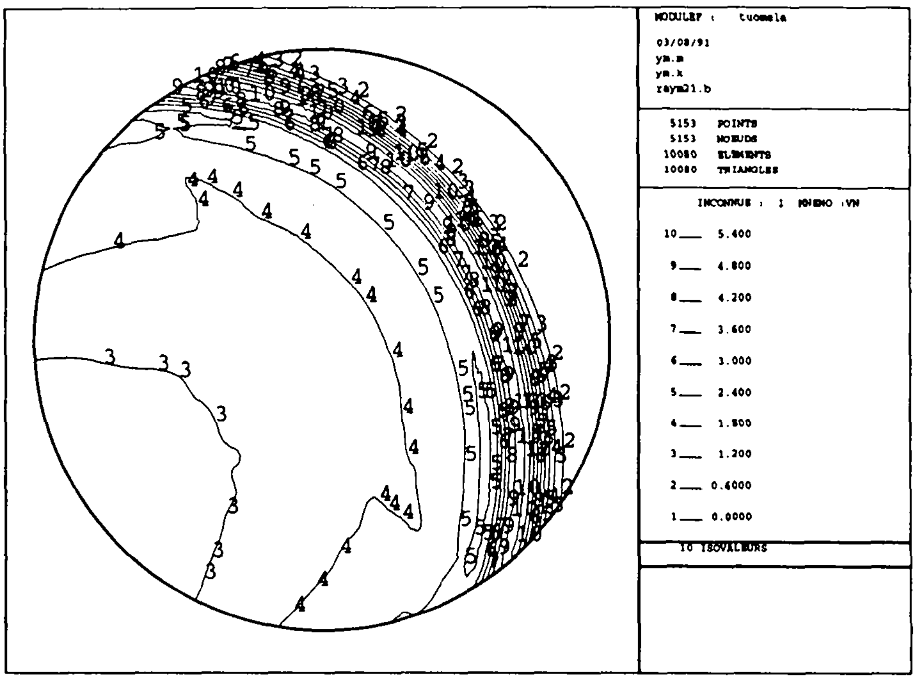

Figure 5.15: Solution at $t = 1.6$ with curvature, second order condition.

L^2	H^1	L^∞	$\sqrt{E}$
0.340	0.336	0.464	0.306
0.194	0.220	0.383	0.282
0.309	0.301	0.540	0.241
0.138	0.148	0.246	0.157

Table 1: Relative error in various norms; first test case: $h = 0.278$, $\delta t = 0.2$, $L = 4$, $N = 14.4$ and $t = 15$.

Using convolution one can in fact calculate the exact solution and in tables 1, 2 and 3 we show the relative errors in different tests using various norms. The first row corresponds to the case 'without curvature, first order', the second 'with curvature, first order', the third 'without curvature, second order' and the fourth 'with curvature, second order'. In the first column there is the L^2 norm, in the second H^1 norm, in the third L^∞ norm and in the fourth the energy norm. We recall that the natural energy for the wave equation is

$$E(u) = \int_\Omega |u_t|^2 + |\nabla u|^2$$

and the energy norm is then the square root of E. The discrete energy was calculated from the formula

$$E_h(u_h) = \frac{1}{\delta t^2}\langle u_h^{n+1} - u_h^n, M(u_h^{n+1} - u_h^n)\rangle + \frac{1}{4}\langle u_h^{n+1} + u_h^n, R(u_h^{n+1} + u_h^n)\rangle$$

The importance of the curvature term and second order term is clear.

Then taking the domain shown in figure 5.16 we have a different kind of problem: one sends the signal from the left using a suitable Dirichlet boundary condition, on horizontal parts of the boundary we take the homogeneous Neumann boundary conditions and on the right the absorbing boundary conditions. This kind of situation might occur when modelling the diffraction from the periodic structures (see [UM] for such a problem in the time harmonic case).

At the Dirichlet boundary we use the same g as before (or rather $g/50$), and the domain is the 'unit square with the triangle added'. In particular we took zero initial conditions, $h = 0.02$, $\delta t = 0.0125$ and $L = 0.4$. Then in figures 5.17, 5.18, 5.19 and 5.20 we have the solutions at $t = 1.6$ and in figures 5.21, 5.22, 5.23 and 5.24 the corresponding solutions at $t = 2.6$.

L^2	H^1	L^∞	$\sqrt{E}$
0.202	0.276	0.353	0.255
0.173	0.293	0.643	0.260
0.123	0.216	0.368	0.197
0.055	0.163	0.144	0.149

Table 2: Relative error in various norms; second test case: $h = 0.02$, $\delta t = 0.0125$, $L = 0.3$, $N = 15$ and $t = 0.9$.

L^2	H^1	L^∞	$\sqrt{E}$
0.276	0.386	0.290	0.525
0.159	0.335	0.328	0.520
0.213	0.338	0.229	0.489
0.121	0.302	0.188	0.480

Table 3: Relative error in various norms; third test case: $h = 0.25$, $\delta t = 0.1$, $L = 3$, $N = 12$ and $t = 15$.

L^2	H^1	L^∞	$\sqrt{E}$
0.126	0.143	0.154	0.143
0.107	0.137	0.144	0.138
0.130	0.093	0.215	0.089
0.059	0.049	0.076	0.047

Table 4: Remaining norms divided by maximum norms.

Finally we present some quantitative information on errors. In this case we do not need to calculate the exact solution: at $t = 2.6$ it should be identically zero. So all that remains is undesirable and in table 4 there are the norms of the numerical solution divided by the maximum value of the corresponding norms (that is, the norm that was calculated when the whole signal was already sent and it had not yet reached the absorbing boundary). In general the results are as expected from previous tests; note in particular that with the second order conditions without curvature there has been rather a strong reflection from the tip of the domain, and that with curvature this is considerably diminished.

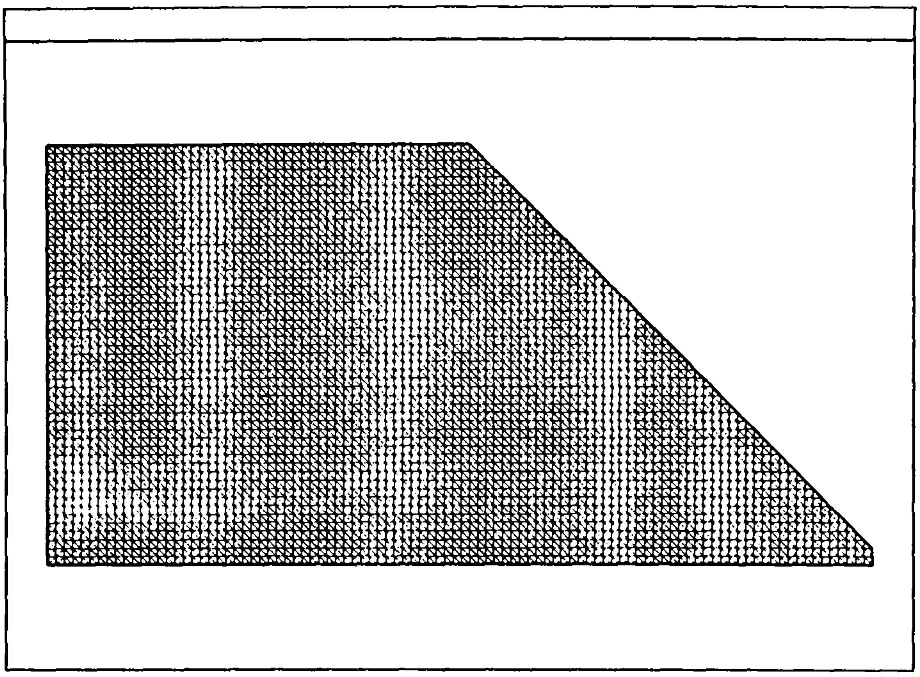

Figure 5.16: Third domain and the mesh used.

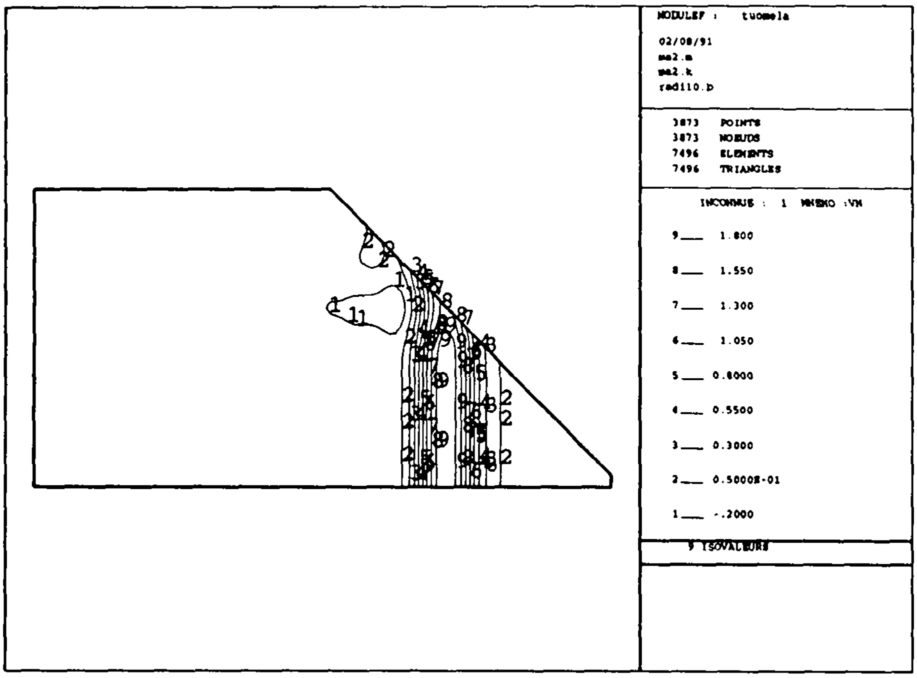

Figure 5.17: Solution at $t = 1.6$ without curvature, first order condition.

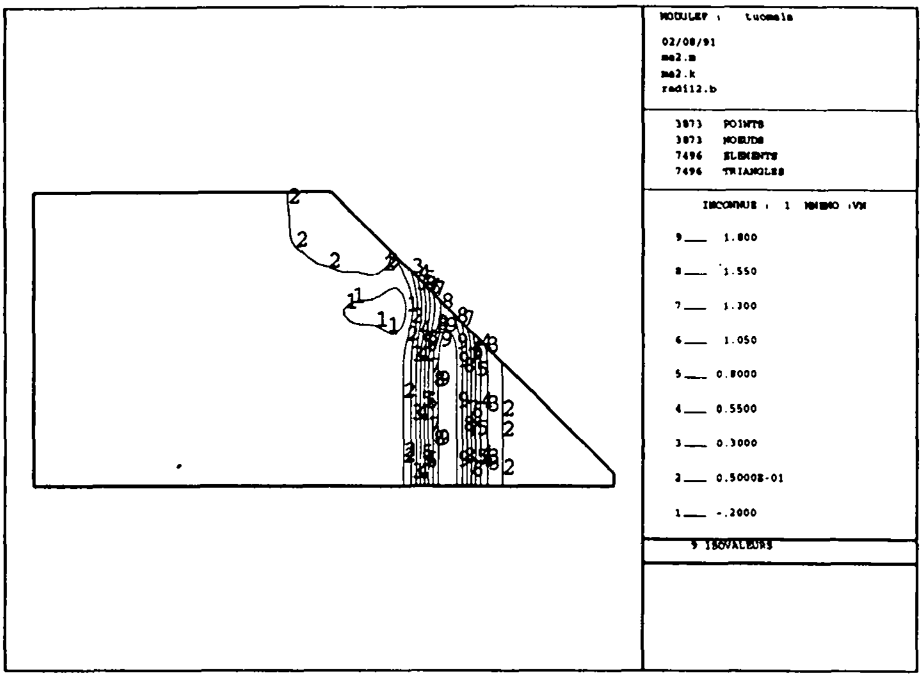

Figure 5.18: Solution at $t = 1.6$ with curvature, first order condition.

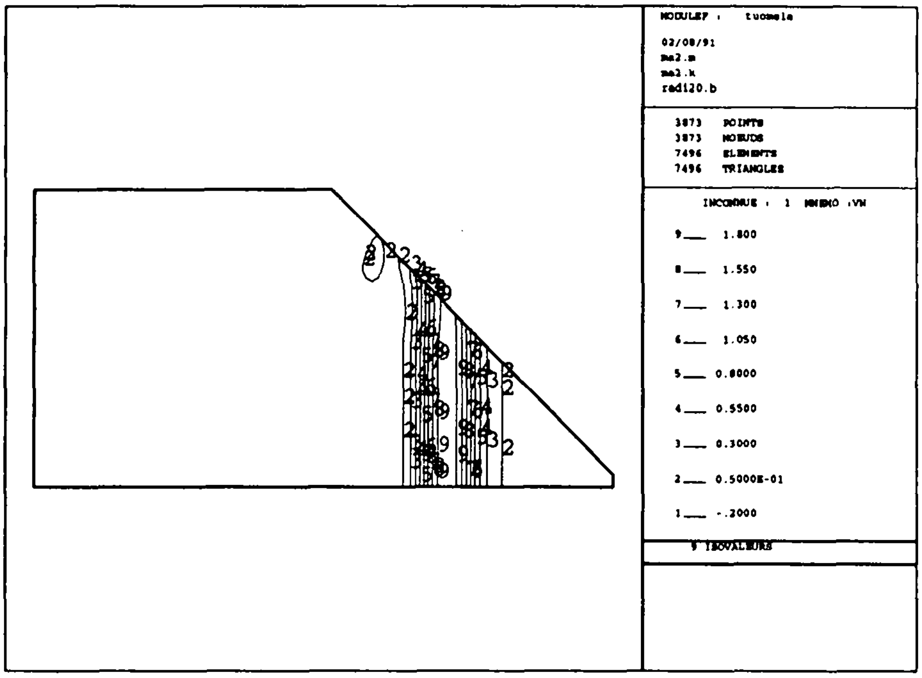

Figure 5.19: Solution at $t = 1.6$ without curvature, second order condition.

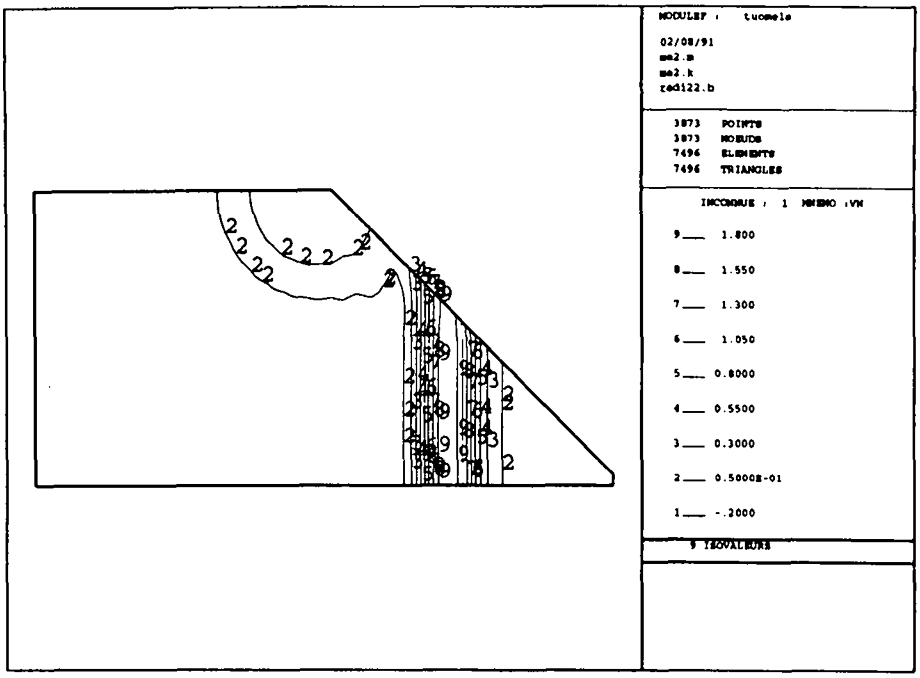

Figure 5.20: Solution at $t = 1.6$ with curvature, second order condition.

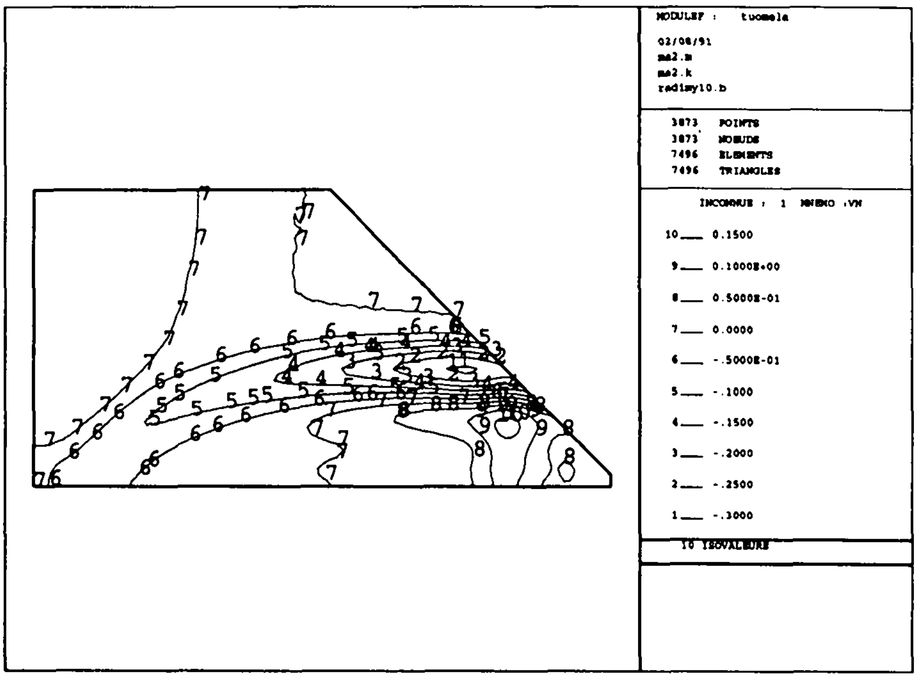

Figure 5.21: Solution at $t = 2.6$ without curvature, first order condition.

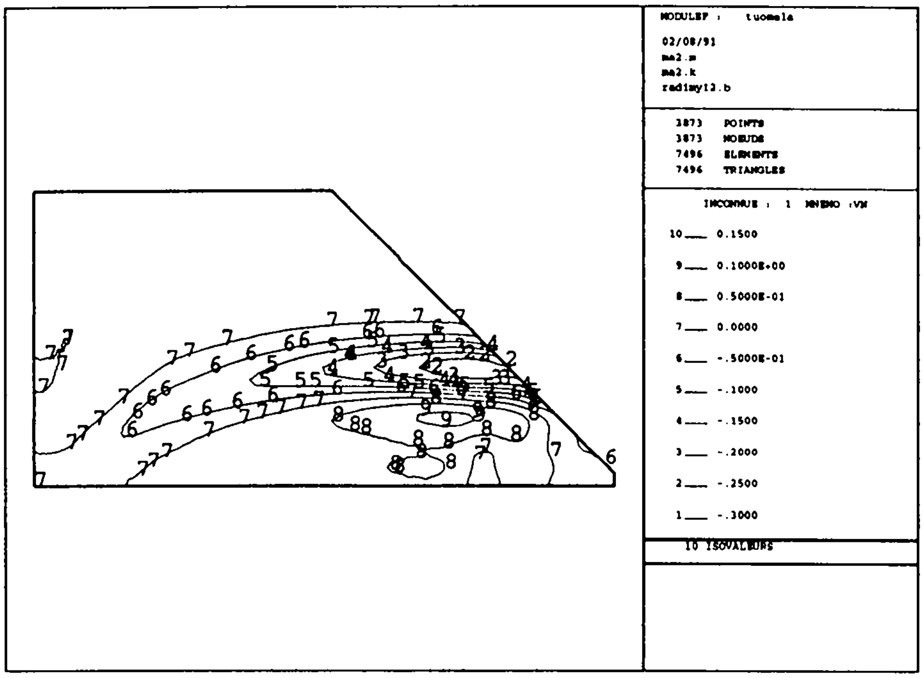

Figure 5.22: Solution at $t = 2.6$ with curvature, first order condition.

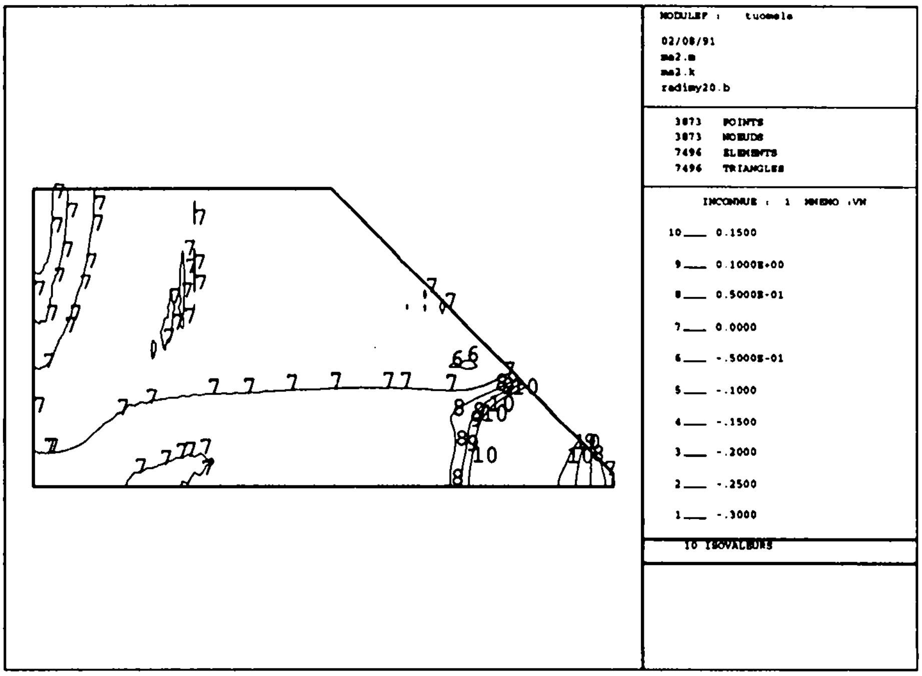

Figure 5.23: Solution at $t = 2.6$ without curvature, second order condition.

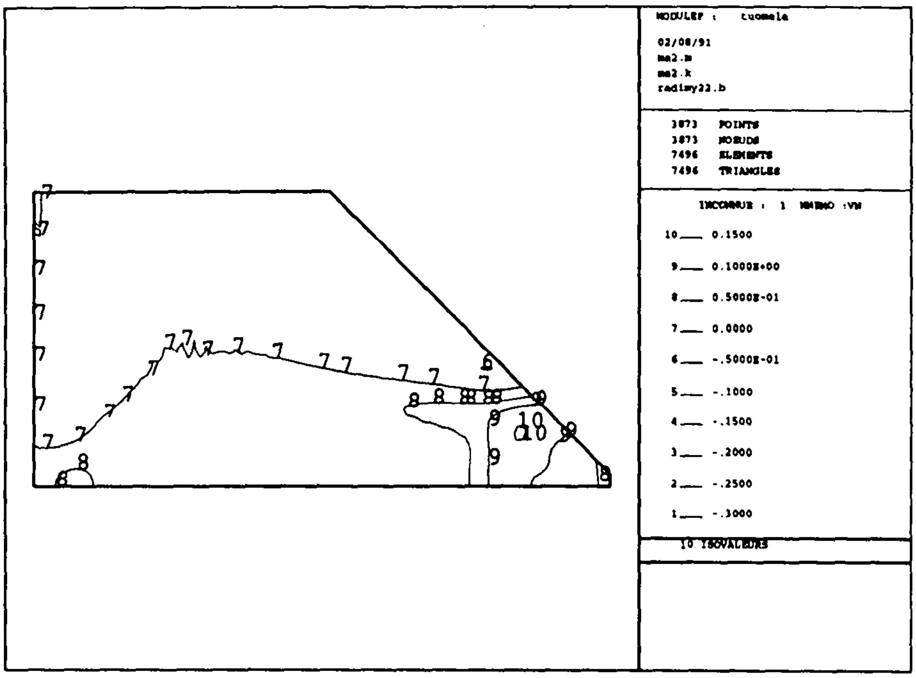

Figure 5.24: Solution at $t = 2.6$ with curvature, second order condition.

6 Conclusion

We have considered the implementation of the absorbing boundary conditions in the finite element context. It was seen that this raises different kinds of problems than finite difference implementation. On the other hand it also offers new solutions: by taking into account the curvature of the boundary we can very simply treat any boundaries and at the same time get good corner conditions. Although we only made numerical experiments in two dimensions we think that the conclusions remain valid also in three dimensions.

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